EXERCISES

Exercise 1. A topological space X is called *sequentially compact* if every sequence of points in X has a convergent subsequence converging to a point in X. Let (F_n) be a sequence of non-empty closed subsets of a sequentially compact set X such that $F_n \supset F_{n+1}$ for all $n = 1, 2, \ldots$, then $\bigcap_{n=1}^{\infty} F_n \neq \emptyset$.

Exercise 2. Let X be a subset of an Hilbert space H which is sequentially compact. Show (using Exercise 1) that X is compact.

Exercise 3. Let (π, V) be an admissible representation. Prove that its contragredient $(\widetilde{\pi}, \widetilde{V})$ is admissible, and that $\widetilde{\widetilde{\pi}} \simeq \pi$.

Exercise 4. Let V be a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ and K a compact open subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Show that every vector in a smooth representation of $\mathrm{SL}_2(\mathbb{Q}_p)$ is K-finite.

Exercise 5. Let (π, V) be a smooth representations of $\mathrm{SL}_2(\mathbb{Q}_p)$. Let K be an open compact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Show that V is K-semisimple.

Exercise 6. Let T be the diagonal torus in $\mathrm{GL}_2(\mathbb{Q}_p)$ and $T_0 := T \cap \mathrm{GL}_2(\mathbb{Z}_p)$. Compute the spherical Hecke algebra $\mathcal{H}(T, T_0)$, which is defined to be the convolution algebra of bi-T_0-invariant locally constant functions on T.

Exercise 7. Let G be a p-adic group and let K_0 be a maximal compact subgroup of G. Let I_χ be an unramified principal series representation of G with respect to K_0 that is not generated by its unique (up to constant multiples) spherical vector f. Show that there is a non-zero intertwining operator $T \colon \mathrm{I}_\chi \to \mathrm{I}_{\chi'}$ from I_χ to another unramified principal series $\mathrm{I}_{\chi'}$ such that $Tf = 0$.

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