Exercise sheet on property (T) (Indira Chatterji)

Most definitions and concepts have been explained in class, the idea is to review them here.

Exercise 1: let π be a unitary representation, and assume that the trivial representation is in the completion of π in $\tilde{G}$.
Express that condition in terms of almost invariant vectors.

Exercise 2. Let $G = \langle S \rangle$ have property (T), and take $0 < \delta \leq 2$.
Show that $\exists\, \varepsilon > 0$ such that $\forall\, \pi \in V_T(\xi, S, \varepsilon)$, then the invariant vector η for π can be chosen δ-close to ξ.

Sketch: set $\varepsilon = \frac{1}{2}\alpha\delta$ where $\alpha > 0$ is a Kazhdan constant and decompose $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ where $\mathcal{H}_0 = \mathcal{H}^G$ G-invariant vectors show that $\eta = \frac{\xi_0}{\|\xi_0\|}$ does the job.

Exercise 3 (Mautner's lemma) G a topological group, $\pi \in \tilde{G}$
Take $x \in G$ and let (y_i) a sequence in G such that $y_i x y_i^{-1} \longrightarrow e$. If $\pi(y_i)\xi = \xi$ $\forall i$, then $\pi(x)\xi = \xi$.

Exercise 4 : let $N = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \} < SL_2(\mathbb{R}) = G$

For $(\pi, \mathcal{H}) \in \hat{G}$, show that if $\xi \in \mathcal{H}$ is N-invariant then ξ is G-invariant.

Hint for Ex 4 :

(1) For $(\lambda_i) \subseteq \mathbb{R}_+^*$ and $\lambda_i \to 0$ $a_i = \begin{pmatrix} \lambda_i & 0 \\ 0 & \lambda_i^{-1} \end{pmatrix}$

check that $\lim a_i h a_i^{-1} = \lim a_i h' a_i^{-1} = Id$ $\forall h \in N$ $h' \in N^t$

(2) For $a = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \in SL_2\mathbb{R}$ and λ_i as above find $(h_i) \in N$ such that $h_i \begin{pmatrix} 0 & \lambda_i^{-1} \\ \lambda_i & 0 \end{pmatrix} h_i^{-1} \longrightarrow a$

(3) Deduce that ξ is also A-invariant and N^t-invariant and conclude.

Exercise 5 : $H = \{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \mid x \in SL_2\mathbb{R} \} < SL_3\mathbb{R} = G$

Take $(\pi, \mathcal{H}) \in \hat{G}$ and $\xi \in \mathcal{H}$ which is H-invariant and show that ξ is G-invariant as well.

Hint : conjugate elementary matrices by $\begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda^{-1} \end{pmatrix}$ and let $\lambda \to 0$ and use Mautner's lemma.

Exercise 6 : Show that $H = \{ \begin{pmatrix} a & b & y \\ c & d & x \\ 0 & 0 & 1 \end{pmatrix} \in SL_3\mathbb{R} \}$ is isomorphic to $SL_2\mathbb{R} \ltimes \mathbb{R}^2$.

Exercise 7 : Convince yourself that the only $Sl_2\mathbb{R}$-invariant probability measure on the Borel sets of $\mathbb{R}^2$, is the Dirac mass at $\{0\}$.

Exercise 8 : $G = Sl_2\mathbb{R} \ltimes \mathbb{R}^2$ $N = \mathbb{R}^2$, the pair (G,N) has relative property (T): Any $\pi \in \hat{G}$ with almost invariant vectors, has N-invariant vectors.

sketch: (1) Take $(\xi_i) \subseteq \mathcal{H}$ with $\lim \|\pi(g)\xi_i - \xi_i\| = 0$ $\forall g \in G$ show $\varphi_i(g) := \langle \pi(g)\xi_i, \xi_i\rangle \to 1$ uniformly on compacts

(2) Fourier transform $\widehat{\varphi_i|_{\mathbb{R}^2}} := \mu_i$ is a probability measure on $\widehat{\mathbb{R}^2} = \mathbb{R}^2$.

(3) $\lim \mu_i = \mu$ $Sl_2\mathbb{R}$-invariant proba measure on $\mathbb{R}^2$

(4) Deduce that $\mu = \delta_{\{0\}}$ and use the decomposition

Thm: Every unitary representation $(\pi, \mathcal{H})$ of G decomposes as $\int_{\hat{G}} \pi_x \, d\mu(x)$ where π_x are irreducible

to argue that $\delta_{\{0\}}$ creates an invariant vector for π.

Exercise 9 : Read & comment the paper below.

CONNECTION OF THE DUAL SPACE OF A GROUP WITH THE STRUCTURE OF ITS CLOSE SUBGROUPS

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In this paper is investigated the structure of discrete subgroups of a Lie group (real and p-adic) with a finite volume factor space. In particular, it is proved that if $\Gamma \subset G$ is a discrete subgroup of a simple group of rank* greater then two, such that the volume of the factor space G/Γ is finite, then Γ has a finite number of generators and the group $\Gamma/[\Gamma,\Gamma]$ is finite. In the case of real numbers the first theorem gives a positive answer to part of a hypothesis of Zigel' on the finiteness of the number of sides of a fundamental polygon. In this development is used information on the structure of the dual space of the group Γ, the space of its unitary irreducible representations $\hat{\Gamma}$.

The paper consists of three parts. In part 1 it is shown how the structure of $\hat{\Gamma}$ is connected with ordinary properties of Γ. In part 2 is shown how to obtain knowledge of the structure of $\hat{\Gamma}$ from the properties of $\hat{G}$. And, finally, in part 3 is investigated $\hat{G}$ in the case where G is a Lie group of rank greater than two.

1. Let G be a locally compact group. The dual space of the group G (denoted $\hat{G}$) is the set of unitary irreducible representations of this group with a topology. Let us describe this topology; more precisely, let us describe the basis of a neighborhood of any representation. Let there be given the representation $G: g \to T(g)$ in the space L. We select the vector $X \in L$, the compact K in G, and the number $\varepsilon > 0$. Let us say that the representation $g \to T'(g)$, operating in the space L', lies in an (X, K, ε)-neighborhood of T(g) if there exists a vector $Y \in L'$ such that $|(T(g)X, X) - (T'(g)Y, Y)| < \varepsilon$, when $g \in K$. We denote by $\tilde{G}$ the set of all unitary representations with the same topology. Note that in the completion of any representation lies all its subrepresentations.

We say that G possesses the property T if the trivial representation is an open set in $\hat{G}$. The property T is equivalent to another property of G: if the trivial representation lies in the completion of the representation $P \in \tilde{G}$, then it enters there in a linear term.

Any compact group possesses property T. The converse is true only in a weaker sense.

THEOREM 1. If a group possesses property T, then the group G/[G, G] is compact.

Proof. It is evident that if G possesses property T, then its factor-group also possesses property T; further let us apply the duality of Pontryagin. In particular, the group G must be unimodular.

THEOREM 2. Let Γ be a countable discrete group with the property T. Then Γ is a group with a finite number of generators.

Proof. Enumerate the elements of Γ: $\gamma_1, \ldots, \gamma_n, \ldots$. Denote by Γ_n the subgroup of Γ, generated by $\gamma_1, \ldots, \gamma_n$. It must be shown that an n can be found such that $\Gamma_n = \Gamma$. For this it is sufficient to show that for a certain n the subgroup Γ_n has the terminal index in Γ. Assume this is not true. We denote by $T_n(\gamma)$ the representation of Γ induced by the trivial representation of Γ_n. Because the index of Γ_n is without bound in Γ, $T_n(\gamma)$ does not comprise a trivial representation of Γ. But in the space of the representation $T_n(\gamma)$ is a vector which is invariant relative to Γ_n. Any compact K in Γ consists of a finite number of elements and therefore lies in Γ_n when $n > n(K)$. We have arrived at a contradiction, since we obtained that in any (X, K, ε)-neighborhood of a trivial representation of Γ lies $T_n(\gamma)$ when $n > n(K)$.

HYPOTHESIS 1. If Γ is a discrete group having property T, then it is a group with a finite number of correspondences.

*The rank of a group G is the dimension of the maximal decomposed torus (in the real case this is the vector part of the mapped subgroup).

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2. Let us show now how to obtain information about the structure of $\hat{\Gamma}$ from the properties of $\hat{G}$.

THEOREM 3. Let the group G have the property T; Γ is its close unimodular subgroup with a finite volume factor-space. Then Γ has property T.

Proof. Let us examine the transformation $\varphi: \tilde{\Gamma} \to \tilde{G}$, induced in the sense of Frobenius (in the construction a unimodular Γ is used).

Theorem 3 follows directly from two properties of this transformation:

a) φ is continuous;

b) if $P \in \tilde{\Gamma}$ does not comprise a trivial representation of Γ, then $\varphi(P)$ does not comprise a trivial representation of G.

Property a) is derived from the results of the paper [1]; however, in the case under consideration it is easily proved.

Property b) follows at once from the definition of φ.

THEOREM 4. If we have a point sequence of topological groups: $0 \to G' \to G \to G'' \to 0$, where G' and G" possess property T, then G possesses property T.

The proof is evident.

3. We indicate now the class of groups possessing property T.

THEOREM 5. The group Sl (3, K) possesses property T when K is any locally compact field which is not discrete.

The proof of the theorem rests on a number of lemmas. We introduce the notation:

H is a subgroup of the group Sl (3, K) of the form $\begin{pmatrix} a & b & x \\ c & d & y \\ 0 & 0 & 1 \end{pmatrix}$, G', G", S are respectively subgroups of matrices of the form

$$\begin{pmatrix} a & 0 & b \\ 0 & 1 & 0 \\ c & 0 & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & c & d \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

S' is a subgroup of the group Sl (2, K) of the form $\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$.

LEMMA 1. If in a representation of Sl (3, K) we have a vector invariant relative to S, then it is invariant relative to Sl (3, K).

Proof. The corresponding assertion for Sl (2, K) and S' is proved in [2]. Therefore, the corresponding vector will be invariant relative to G' and G", and they generate Sl (3, K).

COROLLARY. The restriction of any nontrivial irreducible representation of Sl (3, K) on H is distributed on irreducible representations of H, nontrivial on S.

LEMMA 2. In the completion in $\tilde{H}$ of a regular representation of $\tilde{H}$ is contained any irreducible representation of H, nontrivial on S.

Lemma 2 is easily proved by the methods of the paper [3].

LEMMA 3. A trivial representation of H does not lie in the completion of a regular representation.

Lemma 3 follows from [2] and from the fact that the restriction of a regular representation of H on Sl (2, K) is multiply regular.

Theorem 5 follows at once from Lemmas 1-3.

THEOREM 6. Let G be a simple algebraic group on K, containing a subgroup G' such that: a) G' is an algebraic group possessing property T, b) G' and the maximal compact subgroup U generate G. Then G possesses property T.

(For real and decomposable p-adic groups b) follows from a). Apparently, this is always true.)

We prove two lemmas as a preliminary.

LEMMA 4. There exists such a number N that any element from G is distributed in the product over no greater than N elements from G' and U.

Proof. For real and decomposable p-adic groups the lemma follows at once from the decomposition G= UAU, where A is the vector part of the mapped subgroup. (This decomposition is well-known in the real case and for decomposable p-adic groups is proved in [4].) Since the decomposition G= UAU is obviously always correct, then we only project a proof not using this decomposition.

First it is pointed out that there exists a finite number of elements $u_1', \ldots, u_n'; u_1'', \ldots, u_n'' \in U$ such that the subsets $G_i' = u_i'' G u_i''$ generate G. Then the fact that G_i' and G are algebraic and the uncountability of the field K are used.

LEMMA 5. Let G be a locally compact group, T(g) be its invariant representation in the space L, and the vector $X \in L$ be such that $\mathrm{Re}(T(g)X, X) \geq \varepsilon > 0$. Then T(g) comprises a trivial representation.

Proof. It is easily seen that it is sufficient to prove the theorem for the subgroup G with a finite number of generators, i.e.,for the case where G is a free group with generators $g_1, \ldots, g_n$.

Let us indicate an apparent proposition. Let there be given in a Hilbert space L the n subspaces $L_1, \ldots, L_n$. We denote by P_i the orthogonal projections on L_i, and by P the projection on $L_1 \cap L_2 \cap \ldots \cap L_n$. Then it happens that $(P_1 \ldots P_{n-1} P_n P_{n-1} \ldots P_1)^k \to P$ when $k \to \infty$ in the strong operator topology.

We return to the proof of the lemma for the free group. Assume that there does not exist a vector invariant relative to $T(g_i)$. Using only that formulated proposition, one may easily construct a sequence of positive functions f_n such that a) $f_n \in L^1(G)$, b) $|f_n|_1 = 1$, c) $T(f_n)$ strongly converges to the projection on $L_1 \cap \ldots \cap L_n = 0$. Here as always $T(f_n)$ denotes the operator $\sum\limits_g f(g) T(g)$. But from the conditions of the lemma and from the properties a) and b) of functions f_n it follows that $\mathrm{Re}(T(f_n) X, X) \geq \varepsilon$, and from the property c) it follows that $T(f_n)X \to 0$. The contradiction is obtained and Lemma 5 is proved.

Let us return to the proof of Theorem 6. We examine the representation $P \in \tilde{G}$ in the space L, in the completion of which lies the trivial representation G. Then there exists a sequence of vectors $X_n \in L$ such that $(T(g)X_n, X_n) \to 1$ on any compact. We prove that $(T(g)X_n, X_n) \to 1$ uniformly in G. Then the theorem will follow from Lemma 5.

Let us denote by $L_U^0(L_{G'}^0)$ the subspace of vectors invariant relative to U(G'), and by $L_U^\perp(L_{G'}^\perp)$ the corresponding orthogonal complements. From the property T for U(G') it follows that the projection of X_n on $L_U^\perp(L_{G'}^\perp)$ converges strongly to zero. Using Lemma 4 and the fact that the representation is unitary, we obtain the uniform convergence of $(T(g)X_n, X_n)$ to unity.

Thus, in particular, it is shown that if $\Gamma \subset G$ is a discrete subgroup of a simple real group G of rank greater than two, such that the volume of the factor-space G/Γ is finite, then $H_1(U \backslash G / \Gamma, Z)$ is finite and $\pi_1(U \backslash G / \Gamma)$ has a finite number of generators.

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