Ghent exercises

May 26, 2023

1. For a $(k+1)$-regular graph finish the proof that

$$\mathrm{Spec}(B) = \left\{ \frac{\lambda \pm \sqrt{\lambda^2 - 4k}}{2} \,\middle|\, \lambda \in \mathrm{Spec}\,(A) \right\} \cup \{\pm 1\}^{|E|-|V|}.$$

2. Let $f\colon V \to \mathbb{R}$ with $Af = \lambda f$, and f^t, f^h as defined in the course.
   (a) Show that f^t, f^h are independent iff $\lambda \neq \pm (k+1)$.
   (b) Show that if $f \perp g\colon V \to \mathbb{C}$ then $\{f^h, f^t\} \perp \{g^h, g^t\}$. Compute the Gram matrix of f^h, f^t.
   (c) Show that if $\lambda \in \left[-2\sqrt{k}, 2\sqrt{k}\right]$ then $\mu f^h - f^t, -f^h + \mu f^t$ is an orthonormal basis for Span $\{f^h, f^t\}$, up to scaling.

3. Finish the proof that for any $\varepsilon > 0$, NBRW on the edges of a $k+1$-regular Ramanujan graph with n edges has L^1 mixing time $t_\varepsilon < \log_k n + 3 \log_k \log_k n$ for n large enough. (In fact, 3 can be replaced by any $\delta > 2$).

4. For a prime p with $p \equiv 3 \pmod 4$, the Paley digraph $\mathcal{P}(p)$ has $V = \mathbb{F}_p$ and

$$E = \left\{ a \to b \,\middle|\, \left(\tfrac{b-a}{p}\right) = 1 \right\},$$

where $\left(\frac{\cdot}{\cdot}\right)$ is the Legendre symbol. Show that $\mathcal{P}(p)$ is a $k = \frac{p-1}{2}$-regular digraph, with nontrivial eigenvalues $\frac{-1 \pm i\sqrt{p}}{2}$ (in particular, it is a Ramanujan digraph).

5. Let $G = PGL_d(\mathbb{Q}_p)$, I the Iwahori group $\begin{pmatrix} \mathbb{Z}_p^\times & \mathbb{Z}_p & \cdots & \mathbb{Z}_p \\ p\mathbb{Z}_p & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbb{Z}_p \\ p\mathbb{Z}_p & \cdots & p\mathbb{Z}_p & \mathbb{Z}_p^\times \end{pmatrix}$, P the Borel group of upper triangular matrices in G, and $W = Sym(d) \leq G$ the spherical Weyl group.
   (a) Show that $G = \bigsqcup_{w \in W} PwI$.
   (b) Deduce that for a character χ of P, $\dim \left(Ind_P^G \chi\right)^I \leq d!$ (recall that any Iwahori spherical irreducible representation of G embeds in $\left(Ind_P^G \chi\right)$ for some unramified χ).
   (c) Show that $\dim \left(Ind_P^G \chi\right)^I = d!$ for unramified χ.

6. (a) Show that an r-normal, k-regular Ramanujan digraph $\mathcal{D}$ satisfies

$$\left\| A_{\mathcal{D}}^{\ell}\big|_{\mathbf{1}^\perp} \right\|_2 \leq \binom{\ell + r - 1}{r - 1} \sqrt{k}^{\ell + r - 1}$$

   (or any bound of the form $O\left(\sqrt{k}^{\ell} \cdot poly(\ell)\right)$).
   (b) Deduce that an almost-normal family of Ramanujan digraphs has cutoff.

7. Show that geodesic flow in dimension $j \geq 1$ on the building of $PGL_d(\mathbb{Q}_p)$ is collision-free.

8. Show that $PGU_2(\mathbb{F}) \cong \mathbb{H}(\mathbb{F})^\times / \mathbb{F}^\times$ (Hint: Hilbert '90).

9. Show (explicitly!) that if $\sqrt{-1} \in \mathbb{F}$ then $PGU_d(\mathbb{F}) \cong PGL_d(\mathbb{F}) \cong PU_d(\mathbb{F})$.

10. (a) Let Λ be a group acting on a tree T, and $S \leq \Lambda$ a symmetric generating set ($S^{-1} = S$, $\langle S \rangle = \Lambda$) which takes some vertex v_0 once to each neighbor. Show that Λ acts simply transitively on all vertices.
    (b) Show this is not necessarily true in higher dimensions (and also in a tree, when S is not symmetric).