Some exercices about lattices in p-adic groups

François Thilmany

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Exercise 1 (A tree lattice). Show that the free group F_2 of rank 2 is a cocompact lattice in $\mathrm{Aut}(T_4)$ endowed with the compact-open topology.

Exercise 2 (Finite extension of a lattice). Let Γ be a lattice in a locally compact Hausdorff group G and let Γ' be a discrete subgroup of G containing Γ. Show that Γ' is a lattice, and deduce that the index $[\Gamma' : \Gamma]$ is finite.

Exercise 3 (Connected normalizer of discrete group). Let Γ be a discrete subgroup of a Hausdorff group G. Show that $\mathrm{N}_G(\Gamma)^\circ = \mathrm{C}_G(\Gamma)^\circ$. ($H^\circ$ denotes the connected component of H containing the identity.) Deduce that if additionally Γ is normal and G is connected, then Γ is central.

Exercise 4 (Groups with lattices are unimodular). Let G be a locally compact Hausdorff group and suppose that there is a lattice Γ in G. Prove that G is unimodular, i.e. that the modulus of the action of G on any of its Haar measures is trivial.

Exercise 5 (Lattice remains so after quotient with compact kernel). Let Γ be a discrete subgroup of a locally compact Hausdorff group G and let K be a compact normal subgroup of G. Show that Γ is a lattice in G if and only if the image of Γ is a lattice in G/K.

Exercise 6 ($\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$). Show by reducing matrices that $\mathrm{SL}_2(\mathbb{Z})$ is a non-cocompact lattice in $\mathrm{SL}_2(\mathbb{R})$.

Exercise 7 ($\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$ bis). Compute the covolume of $\mathrm{SL}_2(\mathbb{Z})$ in $\mathrm{SL}_2(\mathbb{R})$. Compare with the estimate from the previous exercise.

Exercise 8. For which prime p is $\mathrm{SL}_2(\mathbb{Z})$ a lattice in $\mathrm{SL}_2(\mathbb{Q}_p)$?

Exercise 9 (An S-arithmetic lattice). Show (without using B&H-C's theorem) that $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$ is a non-uniform lattice in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{Q}_p)$.

Exercise 10 (Non-cocompact lattice in tdlc group). Can a totally disconnected locally compact group admit a non-cocompact lattice?

Exercise 11 (Strong approximation for SL_n). Use the Chinese remainder theorem to prove strong approximation by hand for SL_n over $\mathbb{Q}$ with respect to $R = \{\infty\}$.

Exercise 12 (Failure of strong approximation). Show that strong approximation fails for GL_n and PGL_n over $\mathbb{Q}$.

Exercise 13. Let K be a field of characteristic 0. Prove by hand that $\mathrm{SL}_n(\mathbb{Z})$ is Zariski-dense in $\mathrm{SL}_n(K)$.

Exercise 14 (Endomorphism algebra of lattice)**.** Let G be a semisimple algebraic group over a local field k, and suppose $G(k)$ has no compact factors. Let $\rho : G \to \mathrm{GL}(V)$ be an algebraic representation. Let Γ be a lattice in $G(k)$. Prove that if a subspace W of V is stable under Γ, then it is stable under $G(k)$. Deduce that $k\rho(\Gamma) = k\rho(G(k))$ as sub-k-algebras of $\mathrm{End}(V)$.

Exercise 15 (Centralizer of lattice is central)**.** Let G be a semisimple algebraic group over a local field k, and let Γ be a lattice in $G(k)$. Show that if $G(k)$ has no compact factors, then the centralizer $\mathrm{C}_{G(k)}(\Gamma)$ of Γ is actually central in $G(k)$ (hence finite). Deduce that in presence of compact factors, the centralizer $\mathrm{C}_{G(k)}(\Gamma)$ is a compact subgroup of $G(k)$. Show in this last situation that $\mathrm{C}_{G(k)}(\Gamma)$ need not be central (nor even normal) but could be.

Exercise 16 (Normalizer of a lattice is a lattice)**.** Let G be a semisimple algebraic group over a local field k, and let Γ be a lattice in $G(k)$. Suppose that $G(k)$ is without compact factors. Prove that $\mathrm{N}_{G(k)}(\Gamma)$ is also a lattice. In consequence, $\mathrm{N}_{G(k)}(\Gamma)/\Gamma$ is a finite group.

Exercise 17 (Topology of p-adic group determined by its building)**.** For T a simplicial complex, endow $\mathrm{Aut}(T)$ with the compact-open topology. Show that the p-adic topology on $\mathrm{PGL}_n(\mathbb{Q}_p)$ coincides with the topology induced by the inclusion of $\mathrm{PGL}_n(\mathbb{Q}_p)$ in the automorphism group $\mathrm{Aut}(T)$ of its Bruhat-Tits building T. Does this also hold for $\mathrm{SL}_n(\mathbb{Q}_p)$?

Exercise 18. Construct a lattice in $\mathrm{SL}_2(\mathbb{Q}_p)$ geometrically.

Exercise 19. Construct a lattice in $\mathrm{SL}_2(\mathbb{Q}_p)$ algebraically.

Exercise 20. Compute the covolume of the lattices constructed in the previous exercises.

Exercise 21 (Amalgam for $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$)**.** Show that $\mathrm{SL}_2(\mathbb{Z}[p^{-1}])$ is isomorphic to the amalgamated product $\mathrm{SL}_2(\mathbb{Z}) *_I \mathrm{SL}_2(\mathbb{Z})$, where $I = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, a, b, d \in \mathbb{Z}, c \in p\mathbb{Z} \right\}$. Deduce that it is finitely generated, and compare with the additive group of $\mathbb{Z}[p^{-1}]$.

Exercise 22 (Cocompact lattices are finitely generated)**.** Let k be a non-archimedean local field and Γ a cocompact lattice in $\mathrm{SL}_2(k)$. Prove that Γ is finitely generated.

Exercise 23 (Ihara's theorem)**.** Let Γ be a torsion-free, discrete subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. Prove that Γ is free. Deduce that lattices in $\mathrm{SL}_2(\mathbb{Q}_p)$ are virtually free (of finite rank), hence never have property (T).