Exercices on the mini-course *Ramanujan graphs*

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1. Let X be a k-regular graph on n vertices, $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}$ its eigenvalues. Prove that

   - For every i we have $|\mu_i| \leq k$.
   - The multiplicity of k is equal to the number of connected components of X.

2. Let X be a connected k-regular graph on n vertices. Prove that X is bipartite if and only if the spectrum of X is symmetric about 0, if and only if $\mu_{n-1} = -k$.

3. Let X be a finite graph on n vertices, with eigenvalues $\mu_0 \geq \mu_1 \geq \ldots \geq \mu_{n-1}$. Show that $\sum_{i=1}^{n} \mu_i = 0$, that $\sum_{i=1}^{n} \mu_i^2$ is twice the number of edges, and that $\sum_{i=1}^{n} \mu_i^3$ is 6 times the number of triangles in X.

4. Let $X = (V, E)$ be a finite, connected, k-regular graph. Choose an orientation on X so that each edge e has an origin e^- and an extremity e^+. For every $x \in V, e \in E$, define

   $$\delta(x,e) = \begin{cases} 1 & if \quad x = e^+ \\ -1 & if \quad x = e^- \\ 0 & otherwise \end{cases}$$

   Define $d : \mathbb{R}^V \to \mathbb{R}^E : f \mapsto df$ with $df(e) = \sum_{x \in V} \delta(x,e) f(x)$, and $d^* : \mathbb{R}^E \to \mathbb{R}^V : g \mapsto d^* g$ with $d^* g(x) = \sum_{e \in E} \delta(x,e) g(e)$. Check that d, d^* are adjoint of each other with respect to natural scalar products. Prove that $d^* d = \Delta$, where Δ is the *combinatorial Laplace operator* given by $\Delta = k \cdot Id - A$.

5. The girth of a connected graph X is the length $g(X)$ of its shortest circuit. Let X be a finite, connected, k-regular graph. Observing that, for $r < \frac{g(X)}{2}$, any ball of radius r in X is isomorphic (as a graph) to a ball of radius r in the k-regular tree T_k, deduce the *Moore bound*:

   $$g(X) \leq (2 + o(1)) \log_{k-1} |X|$$

(where $o(1)$ is a quantity that goes to 0 as $|X| \to +\infty$).

6. (link with Indira Chatterji's course on Kazhdan's property (T)) Let Γ be countable group with Kathdan's property (T), let S be a finite symmetric generating set of Γ. Let $(\Gamma_n)_{n>0}$ be a family of finite quotients of Γ, with $|\Gamma/\Gamma_n| \to +\infty$. Show that the family of Cayley graphs $X_n = \mathcal{G}(\Gamma/\Gamma_n, S)$ is a family of expanders. For $F \subset X_n$, you may consider e.g. applying the Laplace operator to the function f_F defined by

$$f_F(x) = \begin{cases} |X_n \setminus F| & if \quad x \in F \\ -|F| & if \quad x \notin F \end{cases}$$

7. A subset F in a finite graph X is *independent* if $A_{xy} = 0$ for every $x, y \in F$. The *independence number* $i(X)$ is the cardinality of the largest independent set. The *chromatic number* $\chi(X)$ is the minimal number of colors necessary to paint the vertices of X so that two adjacent vertices do not have the same color (so that X is bipartite iff $\chi(X) = 2$). Observe that the set of vertices of a given color in a coloring, is independent; deduce that $|X| \leq i(X)\chi(X)$.

8. Let X be a finite, connected, k-regular graph on n vertices. Prove that $i(X) \leq \frac{n}{k} \max\{|\mu_1|, |\mu_{n-1}|\}$. (Hint: take F a maximal independent subset, apply A to the function f_F in exercise 6). Deduce that, if X is Ramanujan non bipartite, then $\chi(X) \geq \frac{k}{2\sqrt{k-1}}$.