Bruhat-Tits-buildings

Part 1: First examples

① the Fano plane

Given finite field $\mathbb{F}_q$ of order q

$G := GL_3(\mathbb{F}_q)$ or $SL_3(\mathbb{F}_q)$

Let $V = \mathbb{F}_q^3$ 3-dim VS over $\mathbb{F}_q$.

$\mathcal{P} := \{$1-dim subspaces of $V\}$ called points

$\mathcal{L} := \{$2-dim — " — $\}$ called lines

We say a point p is incident to a line l if (as linear subspaces) $p \subset l$.

The incidence graph Δ_q of the projective plane is given as follows:

- vertices correspond to elements of $\mathcal{P} \cup \mathcal{L}$
- a pair $\{p, l\}$ forms an edge iff p is incident with l.

Exercise 1: Prove the following

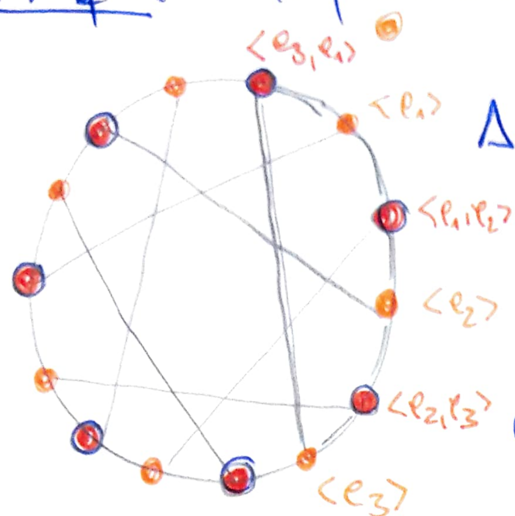
a) $|\mathcal{L}| = |\mathcal{P}| = q^2 + q + 1$

b) Each point is contained in $q+1$ distinct lines.

c) Each line contains $q+1$ distinct points

q^2+q+1

Ex. $q=2$: # points & lines $2^2+2+1=7$



$\Delta_2 \cong$ Heawood graph

$$\mathcal{P} = \{ \underbrace{\langle e_i\rangle}_{3}, \underbrace{\langle e_i + e_j\rangle}_{3,\ i\neq j}, \underbrace{\langle e_1+e_2+e_3\rangle}_{1} \}$$

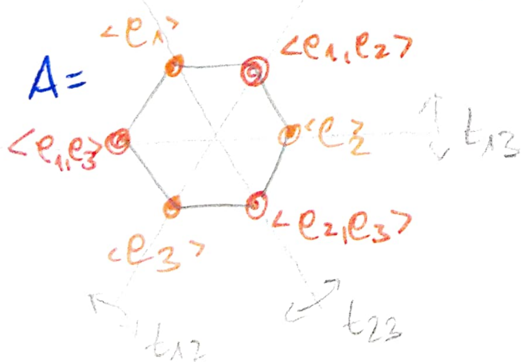
$\mathcal{L}$ = spans of pairs of distinct points

$e_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$

Exercise 2: a) complete the labeling in the picture

b) try to draw Δ_3, i.e. case $q=3$

The graph Δ is a union of hexagons, called <u>apartments</u>. Each of them corresponds to a choice of a basis in V.

$A=$ 

Sym(3) acts on this apartment by permuting the elements of the basis

10 min.

Connection between G and Δ:

As a matrix group $G = GL_3(\mathbb{F}_q)$ acts on $V = \mathbb{F}_q^3$ by multiplication.

$\Rightarrow$ G acts* on Δ and permutes points, respectively lines. (colors are preserved!)

* transitively on pairs of edges $e \subset A$ - apartment/hexagon

Consider $\{e_1, e_2, e_3\}$ std basis

there is an edge $\langle e_1 \rangle \overset{e}{—} \langle e_1, e_2 \rangle$.

What is $\mathrm{stab}(e)$?

Vertex stabilizers:

- $\mathrm{stab}_G(\langle e_1 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in G \right\} =: P_2$
- $\mathrm{stab}_G(\langle e_1, e_2 \rangle) = \left\{ \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\} =: P_1$

Call P_1, P_2 the <u>standard parabolic subgrps</u>.

Then $\cdot$ $\mathrm{stab}_G(e) = P_1 \cap P_2 =: B = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \in G \right\}$

the <u>standard Borel subgrp</u>

<u>Properties A:</u>

a) $\mathcal{L} \underset{1:1}{\hat{=}} G/P_1$, $\mathcal{P} \hat{=} G/P_2$, edges in Δ $\hat{=}$ G/B

i.e. all simplices of Δ are uniquely labeled by a left coset of P_1, P_2 or B.

b) Edges gB and hB share a P_i-vertex iff $gP_i = hP_i$ (or, equivalently, if $g^{-1}h \in P_i$)

Exercise 3: proof these properties.

<u>Stabilizers of A = ⟨ ⟩</u>

Consider again the apartment A wrt the std basis e_1, e_2, e_3.

- pointwise stabilizer is $T = \left\{ \begin{pmatrix} * & & \\ & * & \\ & & * \end{pmatrix} \in G \right\}$
  the <u>torus</u> of G
- setwise stabilizer is $N = \left\{ \begin{matrix} \text{monomial} \\ \text{mxes} \end{matrix} \right\}$

<u>Properties B:</u>

a) N is the normalizer of T in G
b) $N/T \cong \mathrm{Sym}(3)$ in particular this is a Coxeter group!
   ∟: W is generated by the matrixes
   $s_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $s_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$
   satisfying $(s_1 s_2)^3 = \mathbb{1}$.
c) $P_i = B \sqcup Bs_iB$ for $i = 1,2$
d) $G = \bigsqcup_{w \in W} BwB$ the <u>Bruhat-decomposition</u> of G holds

<u>Exercise 4</u>: prove Properties B.

<u>Note</u>: the B, N above is an instance of a <u>(spherical)</u> <u>BN-pair</u>.

local, non-Archim. of char. 0

completion of $\mathbb{Q}$ wrt the q-adic valuation:

$\nu_q\left(\frac{a}{b}\right) = n$ where $\frac{a}{b} = p^n \frac{a'}{b'}$

② A tree for $SL_2(\mathbb{Q}_p)$

one can write elements as $\sum_{n \geq N}' a_n p^n$ with $a_n \in \mathbb{F}_p$

Let K be a field with a discrete valuation $\nu: K^* \to \mathbb{Z}$ s.th.

$\nu(x+y) \geq \inf\{\nu(x), \nu(y)\} \quad \forall x,y \in K$

and $\nu(0) = \infty$.

$\left\{ \sum_{n\geq 0}' a_n p^n \mid a_n \in \mathbb{F}_p \right\}$

- valuation ring $\mathcal{O} := \{x \in K \mid \nu(x) \geq 0\}$ $\quad \mathbb{Z}_p$
- uniformizer π: $\nu(\pi) = 1$. $\quad \pi \cdot \mathcal{O} = \{x \mid \nu(x) > 0\}$, $\pi = p$
- residue field $k := \mathcal{O}/\pi\mathcal{O}$ $\quad = \mathbb{F}_q$

Consider $V := K^2$ 2-dim VS.

A lattice L in V is a f.g. $\mathcal{O}$-submodule of V.

Write $[L]$ for its homothety class. i.e. equivalence up to rescaling with elements in K^*

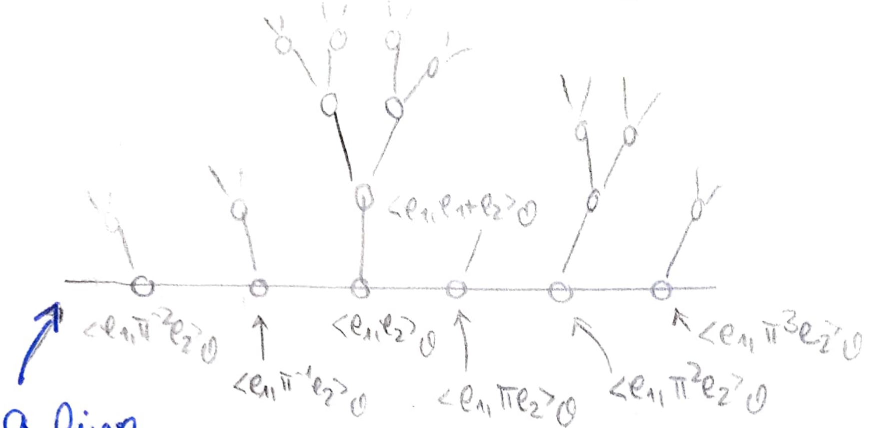
$\rightsquigarrow X :=$ set of classes $[L]$ of lattices in V

We define a graph Δ as follows:

- vertices $\hat{=}$ elements in X
- edges $\underset{[L]}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\underset{[L']}{\circ}$ exist if there exist representatives L, L' s.th. $L' \subset L \subset \pi L'$ (or equiv.: $L' \subset L$ and $L/L' \cong k$).

Thm The graph Δ is a tree without leafs.

[Every pair of vertices (in fact edges) is contained in a common line.



Every pair of lattices L, L' has a common $\mathcal{O}$-basis: $\{b_1, b_2\}$ a basis for L and $\{b_1\pi^a, b_2\pi^b\}$ a basis for L'. $L' \subset L$ iff $a, b \geq 0$.

a line corresponds to a choice of a basis

embedded lines are called apartments

Properties C:

* again transitively on pairs edge $\subset$ apartment

a) $SL_2(\mathbb{K})$ acts* on the tree ~~T~~ Δ

b) A line is setwise stabilized by, again, the semidirect product of $\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rangle =: S$ and translations $T(\mathcal{O}) = \{ \begin{pmatrix} * & \\ & * \end{pmatrix} \in SL_2(\mathcal{O}) \}$

Normaliser of tori $\downarrow$ again N = monomial matrices with entries non-zero in $\mathcal{O}$

$W_0 = N(K)/T(K)$

$W = N(K)/T(\mathcal{O})$ this is a Coxeter group!

c) S reflects the line corresp. to the std basis at the origin $\langle e_1, e_2\rangle_{\mathcal{O}}$

d) Stabilizer of the edge $\langle e_1, e_2\rangle_{\mathcal{O}} \longrightarrow \langle e_1, e_2\pi\rangle_{\mathcal{O}}$ is given by:

$$I = \left\{ \begin{pmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ p\mathbb{Z}_p & \mathbb{Z}_p \end{pmatrix} \in SL_2(\mathbb{Z}_p) \right\} = \text{inverse image of the Borel } B(k) \text{ under the projection } \mathcal{O} \twoheadrightarrow k$$

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} \qquad SL_2(\mathcal{O})$$

~~*~~ For proof see: Serre: Chapter II
Brown: V8B
Thomas: Section 9

30 min

Put $N(K)$ to be the normalizer of the torus $T(K)$, then $(I, N(K))$ is an affine BN-pair for $SL_2(K)$.

→ more this afternoon!

Both ① and ② are examples of buildings!

Formal definition of buildings:

Definition:

A building is a simplicial complex $\Delta \neq \emptyset$ together with a collection of subcomplexes $A \in \mathcal{A}$, called apartments, s.th. the following are satisfied:

(B0) Each apartment is a Coxeter complex

(B1) For any two simplices a, b in Δ there exists an apartment $A \in \mathcal{A}$ s.th. $a, b \in A$.

(B2) if A, A' are two apartments containing a and b then $\exists$ iso $A \to A'$ fixing a & b pointwise.

Exercise 5: a) Prove that the Heawood graph and the tree for $SL_2(\mathbb{Q}_p)$ satisfy (B1) & (B2).

b) Prove that any tree without leafs is a building.

Hint: the line ⋯—o—o—o—o—⋯ is a Coxeter cpx (it is the only euclidean one in dim = 1.)

c) play with Bram Bekker's buildings gallery

③ A little more on Coxeter groups and complexes

A Coxeter group W is a group given by a presentation as follows:

$$W = \langle \overbrace{s_1, \dots, s_n}^{S} \mid s_i^2 = 1\!\!1 = (s_i s_j)^{m_{ij}} \rangle$$

where $m_{ij} \in \{2,3,4,\dots,\infty\}$ $\qquad m_{ii} = 1$

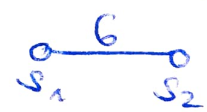
where $m_{ij} = \infty$ means no relation

the Coxeter system (W,S) is encoded by the Coxeter matrix $\mathcal{M} = (m_{ij})_{i,j \in I}$

⇝ encode $\mathcal{M}$ in a graph: vertices ≙ $s_1, \dots, s_n$
edge if $m_{ij} \geq 3$ labeled

Exercise 6:

Prove that the Coxeter groups W_1 and W_2 encoded by $\mathcal{M}_1 = \begin{pmatrix} 1 & 6 \\ 6 & 1 \end{pmatrix}$

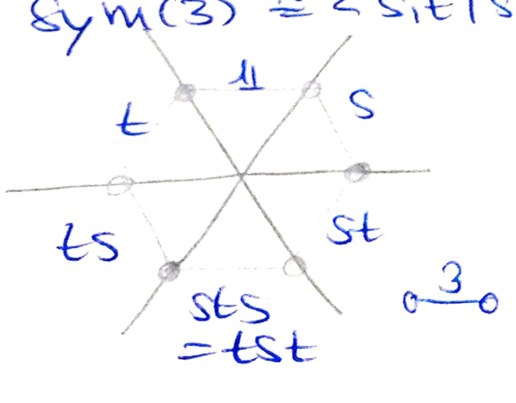


and $\mathcal{M}_2 = \begin{pmatrix} 1 & 3 & \infty \\ 3 & 1 & 2 \\ \infty & 2 & 1 \end{pmatrix}$

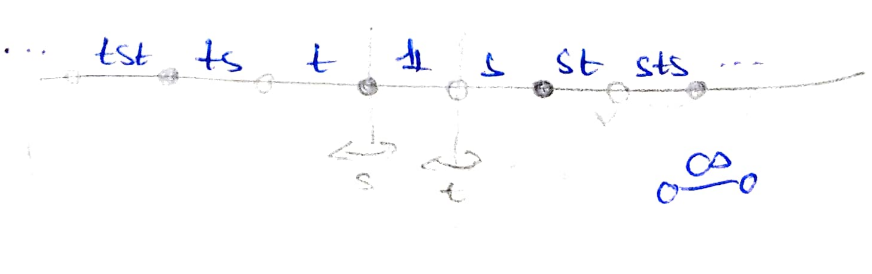
are isomorphic.

Examples:

a) $Sym(3) \cong \langle s,t \mid s^2 = t^2 = (st)^3 \rangle$



b) $D_\infty \cong \langle s,t \mid s^2 = t^2 \rangle \cong \langle t \rangle \ltimes \mathbb{Z}$



40 min

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Properties:

- Coxeter groups come in 3 classes: spherical, affine, infinite but not affine
  these are classified ~> lists of diagrams
- affine Coxeter groups split as semidirect products of a spher. Cox. grp and a translation grp isom. to $\mathbb{Z}^n$, $n = |S|$.
- all Coxeter groups are linear i.e. admit a faithful representation $\rho : W \to GL_n(V)$ where $V = \mathbb{R}$-VS of dim $n = |S|$.
- All Coxeter groups come with a simplicial complex on which they naturally act. (a copy of an apartment!)

Construction of the Coxeter complex:

Given a Coxeter system (W, S) consider

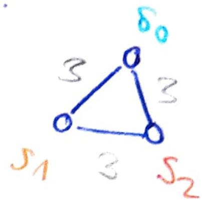
- special subgroups $W_{J'} = \langle S' \rangle < W$
  supported by index set J'
- special cosets $w \langle S' \rangle$, $w \in W$, $S' \subset S$.
  "$\emptyset$ is allowed" $\langle \emptyset \rangle$

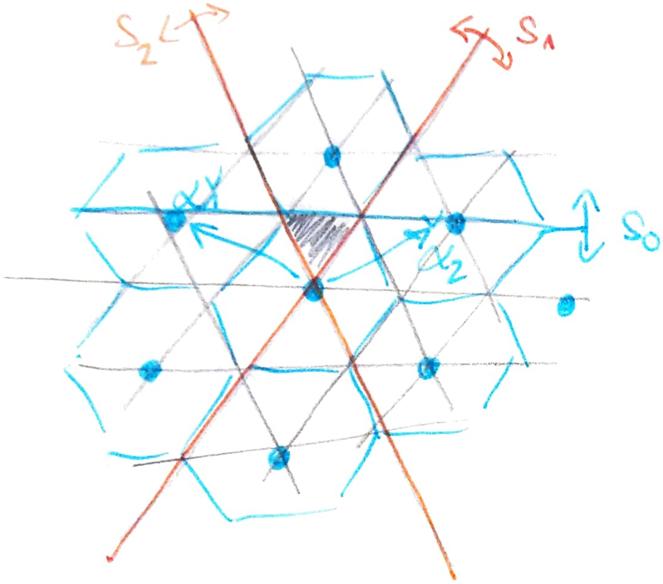
the Coxeter Complex $\Sigma = \Sigma(W, S)$ is the poset of special cosets ordered by reverse inclusion: $B \leq A$ in Σ iff $B \supseteq A$ as subsets of W.

Exercise 7: Construct the Coxeter complexes for the two matrices / diagrams in Exercise 6.

Example Coxeter group of type $\tilde{A}_2$:

$$W = \langle s_0, s_1, s_2 \mid s_i^2,\ (s_i s_j)^3 \ \ i \neq j \rangle$$





• ≙ translation lattice in the Coxeter cplx $\cong \mathbb{R}^2$

$$W \cong \underset{\substack{\uparrow \\ \text{generated by } s_1, s_2}}{W_0} \ltimes \underbrace{\mathbb{Z}^2}_{= T}$$

torus in $SL_3(\mathbb{Q}_p)$

Properties I (see e.g. Brown III)

a) A Coxeter complex is a thin, ~~labelable~~ see c) chamber complex of rank $n = |S|$ on which W naturally acts.

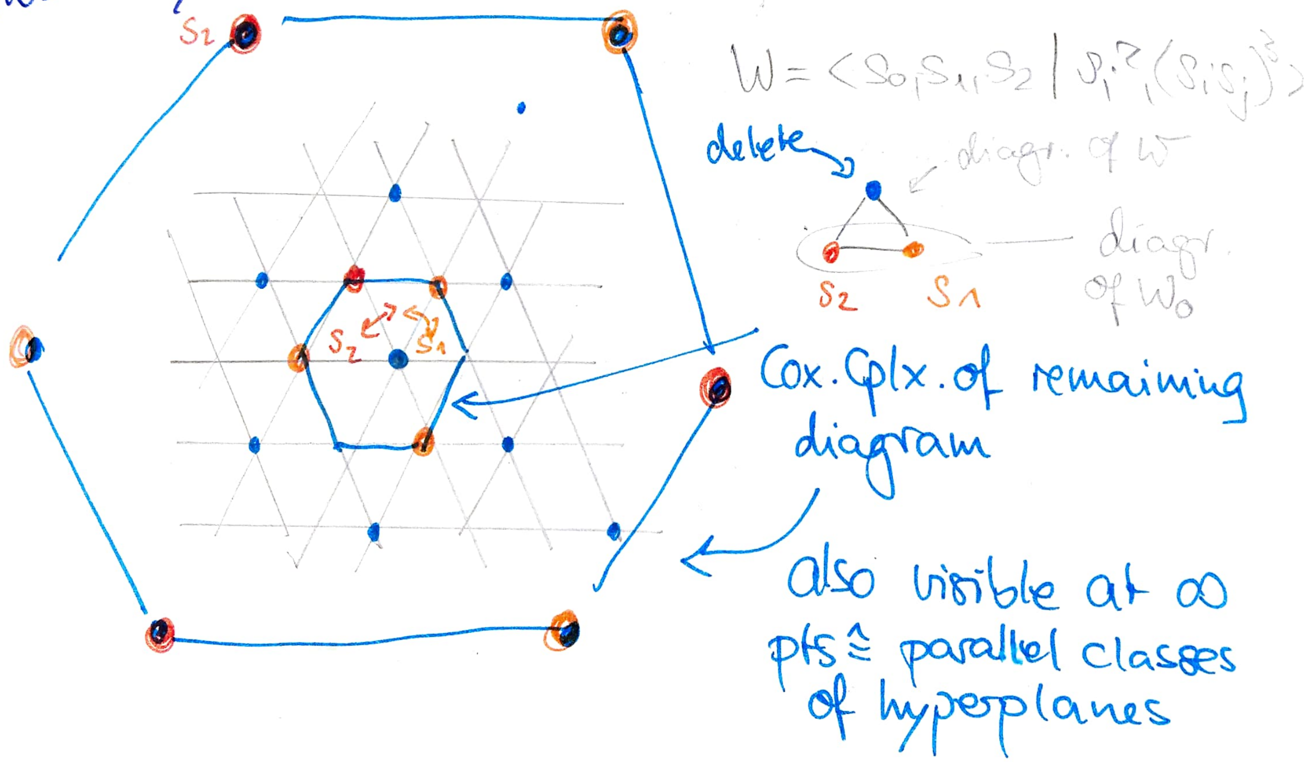
b) Maximal simplices (= chambers or alcoves) are in 1:1 correspondence with elements of W

c) $\overbrace{\text{codim 1 faces}}^{=:\text{ panels}}$ can be colored: at the fund. alcove give the color s_i if the panel is fixed by $s_i \in S$.
Then transfer colors using the W-action

Def. A Coxeter group is <u>affine</u> if the metric realization of its Coxeter complex is $\cong \mathbb{R}^n$ with $n = |S| - 1$.
It is <u>spherical</u> if the Cox. cplx is a sphere S^n (here $n = |S|$).

Fact: Every affine Coxeter group W is the semidirect product of a spherical Cox. grp W_0 of associated type and a translation group isomorphic to $\mathbb{Z}^n$, $n = |S| - 1$.

Two ways to "see" W_0:



$W \cong W_0 \ltimes \mathbb{Z}^n$ here $n=2$

more in Part 3!