Part 2: buildings and BN-pairs

We have already seen the close connection between Sl_n and buildings in examples in Part 1.
We will explore this further now.

Goal: Characterize the buildings Δ in terms of the subgroups B, N, ~~...~~ and vice ~~...~~ versa.

Def. A BN-pair (sometimes called Tits system) for a group G is a pair (B,N) of subgrps of G s.th. the following hold:

(BN0) G is generated by $B \cup N$.

(BN1) $T := B \cap N$ is normal in N

and the group $W := N/T$ has a generating system $S = \{s_1, \dots, s_n\}$ s.th.

(BN2) $\forall\, w \in W$ and $\forall\, s_i \in S$

$$BwB \cdot Bs_iB \subseteq BwB \cup Bws_iB$$

(BN3) $\forall\, s_i \in S: \; s_i B s_i^{-1} \neq B$.

↑ not necessary, corresponds to the bldg being thick

Rmks: (i) S in uniquely determined by (BN0)-(BN3)

(ii) one can prove that W is a Coxeter grp.
see [Brown] V.2A

Exercise 8: Prove that (B,N) in ① and (I,N) in ② are BN-pairs in this sense.

10 min

How to construct a building from a BN-pair?

Thm A (Tits) see e.g. Ronan Thm 5.3

Let G be a group with a BN-pair. (enough to assume (BN0)–(BN2))
Put $W := N/T$ with $T := B \cap N$.
Then there exists a building $\Delta = \Delta(B,N)$ of type (W,S) s.th.

- the set of chambers (= max simpl.) is given by $\{gB \mid g \in G\}$
- two chambers are i-adjacent
  $$gB \sim_i hB \iff g^{-1}h \in P_i := B \sqcup Bs_iB$$
- there is a W-valued distance fct:
  $$\delta(gB, hB) = w \iff g^{-1}h \in BwB.$$

If (BN3) holds, then Δ is <u>thick</u>, i.e. every codim 1 face of a chamber is contained in at least 3 chambers.

Put $c_0 := B$, $A_0 := \{wc_0 \mid w \in W\}$ and let $\mathcal{A} = \{g.A_0 \mid g \in G\}$. Then G acts strongly transitively wrt $\mathcal{A}$ on Δ. The subgrp N stabilizes A_0 setwise.

<u>Exercise 9</u>: Prove that $\forall i$ the set P_i is indeed a subgrp of G. moreover P_i equals its own normalizer

Bruhat decomposition:

From the BN-pair axioms we may prove that G satisfies <u>Bruhat decomposition</u>: 1)

Lemma see eg Brown (chapter V sec 2 ... Ronan ...)

If G has a BN-pair, then

$$G = \bigsqcup_{w \in W} BwB \qquad \text{resp. } G = \bigsqcup_{w\in W} IwI$$

spherical BN-pair — one chamber — all of Δ — one apt

for affine BN-pair

⌐ skip.

Proof (Sketch) ⇝ Exercise 10: (hard) prove this using BN-pair axioms

- Show $G = \bigcup_{w \in W} BwB$. (*) Let $g \in G$.

  By (BN0) $g = b_1 n_1 b_2 n_2 \cdots b_k n_k b_{k+1}$

  $b_i \in B$, $n_i \in N$.

  $\Rightarrow g \in B n_1 B n_2 B \cdots n_k B$

  $= B w_1 B w_2 \cdots w_k B$ with $w_i = n_i T$

  $T = B \cap N$

  repeated application of (BN2) $\Rightarrow$ (*).

- union is disjoint: by induction on $d := \min\{ l_S(w), l_S(w') \}$ where we assume that $w, w' \in W$ are such that $BwB = Bw'B$.

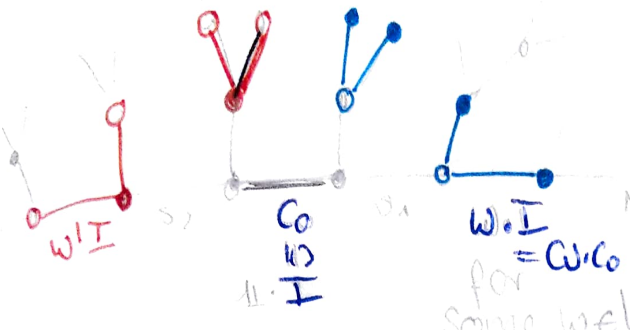
  if $d = 0$ $w = w' = \mathbb{1}$ as $BwB = B$ or $\underset{=0}{d = l_S(w')}$.

  for $d > 0$ argue by contradiction and use (BN2) and induction by writing $w' = s \cdot w''$, $l_S(w'') = l_S(w) - 1$

  see e.g. Thomas p.122

□

Geometric interpretation of Bruhat decomp.



$\hat{=} \{ IwI \}$ has 4 elements for $w = s_1 s_2$

$\hat{=} \; Iw'I$ — " — $w = s_2 s_1$

The double coset IwI (more generally BwB) is the collection of chambers of Δ that are in the same I (resp. B) orbit as $wI \in A_0$.

Bruhat decomp. $\Rightarrow$ $\exists$ exactly one such I-orbit for every chamber in Δ

We will connect (BN2) to galleries tomorrow!

Bruhat decomp. allows to define a

<u>retraction</u>: $\rho_{A_0, C_0} : \Delta \to A_0$

$C \longmapsto w \cdot C_0$

where w is the unique element of W s.th. C is in the IwI cell. (resp. BwB)

<u>Fact</u>: $\rho_{A_0, C_0}^{-1}(w\cancel{B}) = \cancel{BwB}\, . IwI$.

<u>Rmk</u> One can define ρ geometrically using the the simplicial complex definition of a building. $\leadsto$ more on Tuesday in Part III.

From buildings to BN-pairs:

The following is a "converse" to Thm A:

Thm B

Let Δ be a building of type (W,S) and with apartment system $\mathcal{A}$.
Let $G \curvearrowright \Delta$ strongly transitive wrt $\mathcal{A}$.
colorpreserving
Choose fund. chamber C_0 in fund. apmt A_0.
Put $B :=$ pointw. stab in G of C_0
$N :=$ setwise stab. in G of A_0.

Then (B,N) satisfies (BN0) – (BN2).
If Δ is thick, then (BN3) is also satisfied

skip / orally

Remarks

- spherical buildings of rank $= |S| \geq 3$ are classified.
  They all arise from spherical BN-pairs as explained today

  classifying rk $= |S| = 2$, means classifying all projective planes ... open

- affine buildings of dimension ≥ 3 are classified.
  They arise from affine BN-pairs obtained via grps over non-Arch. local fields with discrete valuation

  We call them <u>Bruhat-Tits buildings</u>

the Bruhat–Tits–case:

suppose G is a "semi-simple" matrix grp. over a field K with discrete valuation.

$$\nu : K^* \longrightarrow (\mathbb{Z}, +)$$

Ex. $G = SL_n(\mathbb{Q}_p)$ as in ① of Part I

$$\mathcal{O}/\pi\mathcal{O} = k \longleftarrow \mathcal{O} \hookrightarrow K$$

$$\mathbb{F}_p \cong \ \longleftarrow \mathbb{Z}_p \hookrightarrow \mathbb{Q}_p$$

we then have 3 BN-pairs:

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$

$N =$ monomial mxes

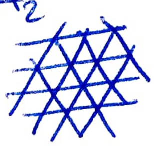
$I = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix} = pr^{-1}(B(k))$

$\underbrace{B(k), N(k)}$ $\downarrow$ **spher. building**

if $n=3$ and $p=2$ we obtain the Heawood graph ⤳ see Part I ①

$\underbrace{I, N(K)}$ $\downarrow$ **affine building**

if $n=3$, $p=2$ apmts are of type $\tilde{A}_2$



around the origin we see the Heawood graph at ∞ the bldg over K

$\underbrace{B(K), N(K)}$ $\downarrow$ **spher. bldg**

if $n=3$, $p=2$ we obtain a bldg of type A_2, apartments are hexagons, but there are ∞-many edges at every vertex