Part 3: more on the Bruhat-Tits-Case

multiple BN-pairs:

① Suppose G is a group over a field K, non-Arch. local with discrete valuation
$\nu: K \to (\mathbb{Z}, +)$

E.g. $G = SL_n(\mathbb{Q}_p)$ with $\nu(\frac{a}{b}) = n$ where $\frac{a}{b} = p^n \frac{a'}{b'}$, a', b' not divisible by p

b) We then have:

$$K \hookleftarrow \mathcal{O} \xrightarrow{pr} k$$

val. ring — residue field

$$\mathbb{Q}_p \hookleftarrow \mathbb{Z}_p \longrightarrow \mathbb{F}_p$$

We then get 3 BN-pairs:

$$\underbrace{B(K), N(K)} \qquad \underbrace{I, N(K)} \qquad \underbrace{B(k), N(k)}$$

$B(*) = \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$

$N(*) =$ mon. mx $/ *$

$I = pr^{-1}(B(k)) = \begin{pmatrix} \mathcal{O} & & \mathcal{O} \\ & \ddots & \\ \pi\mathcal{O} & & \mathcal{O} \end{pmatrix}$

$\downarrow$	$\downarrow$	$\downarrow$
W is finite and Δ_1 spherical	W is infinite Δ_2 affine	W finite Δ_3 spherical
$W \cong Sym(n)$	$W \cong Sym(n) \ltimes \mathbb{Z}^n$	$W \cong Sym(n)$
$n = 3, p = 2$	$n = 3, p = 2$	$n = 3, p = 2$
Δ_1: spherical bldg (locally infinite) apartments $\cong$  at ∞	 apts of Δ_2	Δ_3 = incidence graph of Fano plane apts: 

$$\Delta_1 = \partial\Delta_2 \xleftarrow[\text{at } \infty]{} \Delta_2 \xrightarrow[\text{locally}]{} \Delta_3 = lk_{\Delta_2}(\text{vertex})$$

② The construction of $\Delta(\mathcal{B}, N)$:

Suppose G has a BN-pair.

Put $W := N/T$ where $T = B \cap N$

resp. $I \cap N$

e.g. (B,N) or (I,N) as above

Let S be the generating set, indexed by I

Rmk: One can prove that S is uniquely determined by B and N and consists of all nontrivial $w \in W$ s.th. $B \cup BsB$ is a subgrp. of G.

($\to$ see Brown V, 2B)

Def. Define for a subset $J \subset I$, $W_J := \langle s_j \mid j \in J \rangle$

$\emptyset$ is allowed

and let $P_J := \bigsqcup_{w \in W_J} BwB$

the standard parabolic subgroup of type J.

we used ...

in case (I,N) this is called parahoric

Rmk: $P_{\{i\}}$ is the subgrp P_i we encountered in Part 1

$P_1 = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}$ $P_2 = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix}$

$P_\emptyset = B$. the standard Borel subgrp

resp. std. Iwahori subgrp

Def. A Borel subgrp in G is a conj. of B

A parabolic subgrp in G —″— of some P_J, $J \subset I$.

Properties

a) if $B \le P \le G$ then $P = P_J$ for some $J \subset I$.

b) $P_J \cap P_K = P_{J \cap K}$ and $\langle P_J, P_K \rangle = P_{J \cup K}$

~~c) P_J is the pointw. stabilizer of the J colored simplex in the bldg~~.

Look at the following poset:

$\{ g P_J \mid J \subset I, g \in G \} \leftarrow$ cosets of standard parabolic subgrps

ordered by reverse inclusion.

The realization of the poset is the building $\Delta(B,N)$.

One has:

- max simplices, called chambers $\hat{=}$ cosets $gB = gP_\emptyset$, $g \in G$
- codim 1 faces of chambers $\hat{=}$ cosets $g P_{\{i\}}$, $g \in G$, $i \in I$
- codim k faces of chambers $\hat{=}$ cosets $g P_J$, $g \in G$ $J \subset I$, $|J| = k$

stabilized by $g P_J g^{-1}$

- fund. chamber $\hat{=}$ B

  fund. apartment $\hat{=}$ N-orbit of B

  $\hookrightarrow \{ n \cdot P_J \mid J \subseteq I, n \in N \}$

  poset $\cong \{ w \cdot P_J \mid J \subseteq I, w \in W \}$ here $P_J = B W_J B$

  contains B

  $T = B \cap N$ acts trivially on $P_J \supset B$

  $n = w \cdot T$

  this gives canon. isom. with $\Sigma(W, S)$.

(3) Bruhat decomp. revisited / Retractions

BN-pair axioms imply (see e.g. Thomas p.12)

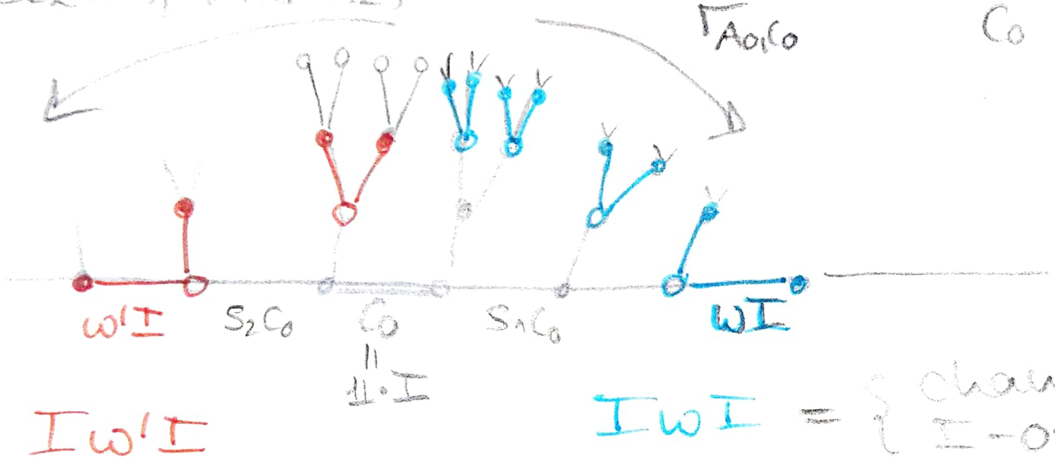
Lemma If G has a BN pair, then

$$G = \bigsqcup_{w \in W_0} BwB \quad \text{resp.} \quad = \bigsqcup_{w \in W} IwI$$

Bruhat decomp.   affine Bruhat decompos.

Geometric interpretation:

$SL_2(K)$, $\nu: K \to \mathbb{Z}$



$C_0 \triangleq I$ affine B / Iwahori subgp

$IwI = \{$chambers in the I-orbit of $wI\}$

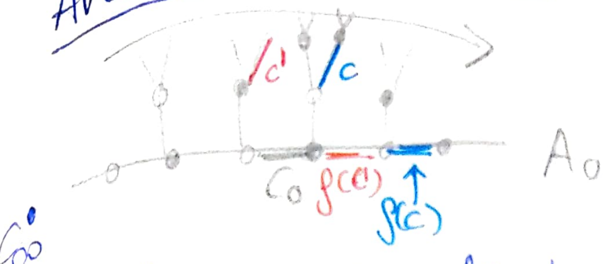
(affine) Bruhat decomp. $\Rightarrow$ $\forall g \in G$ (and hence all gI) there exists exactly one $w \in W$ s.th. $gI \in IwI$

Define a **retraction**

$$r_{A_0,C_0} : \Delta(I, N) \longrightarrow A_0$$
$$gI \longmapsto wI$$

write $gI \in IwI$

Another retraction:



define $\rho_{A_0,\infty} : \Delta \to A_0$ geometrically as follows:

„C_∞" corresponds to "a chamber in $\partial\Delta(I,N)$

i.e. in $\Delta(\mathcal{B}(K),N(K))$

Fact: fix C_∞ at infinity.
For every chamber c $\exists$ Apmt A s.th. A contains c and ∂A contains C_∞. Any such apartment intersects A_0 in a ray pointing towards C_∞.

Put $\rho_{A_0,\infty}(c) := \tilde{c}$ where $\tilde{c}$ is the isometric image of c in A_0 under iso $A \to A_0$ fixing $A \cap A_0$

Algebraically:

$$\rho^{-1}(\underbrace{wI}_{\text{chamber in } A_0}) = U.wI \quad \text{where } U = \mathrm{Stab}_G(C_\infty)$$

$$= \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \quad * \in K$$

Final remark: a BT-building of rk n has n types of (chimney) retractions encoding various orbits / decompositions of G & double coset intersections of various kinds.

The retractions can be understood using gallery combinatorics

Summary of Parts 1 and 2 and 3:

Coxeter grps

Fano plane example

axiomatic definition of a building Δ

SL_2-tree example

via strongly transitive actions ↓

↑ explicit poset constr.

Examples of spherical BN pairs

axiom. defn. of BN-pairs

affine BN pairs

Bruhat decomp. & retractions } Part 3

~~Outlook: (Part 3)~~

~~→ more on the Bruhat-Tits case~~

~~→ more on retractions~~

~~→ and how they connect with combinatorics of galleries in Coxeter groups.~~

$\rightsquigarrow$ Sources:

Brown: Buildings $\rightsquigarrow$ Abramenko-Brown

Thomas: Geom. & topol. aspects of cox. grps & bldgs

Ronan: Lectures on buildings

Schwer: (Survey) Shadows in the wild: folded galleries and their applications

Weiss: Structure of spherical/affine buildings (2 books)

original works of Tits, Bruhat-Tits

Tits: lectures notes on spherical buildings and BN Pairs