# Local representation theory and Hecke algebras I

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# Hilbert spaces I

## Definition

A *Hilbert space* is an inner product space $(H, \langle \ , \ \rangle)$ that is complete with respect to the norm defined by the inner product:

$$\|x\| := \sqrt{\langle x, x\rangle} \quad \text{for } x \in H.$$

## Recollection

A sequence of vectors $(x_n)$ in a normed vector space is called a *Cauchy sequence* if for every $\epsilon > 0$ there exists a number $M$ such that $\|x_m - x_n\| < \epsilon$ for all $m, n > M$. A vector space is said to be *complete* if every Cauchy sequence converges.

## Definition

A Hilbert space is called *separable* if it has a countable orthonormal basis

# Hilbert spaces II

## Definition

Let $H_1$, $H_2$, ... be separable Hilbert spaces. Their Hilbert sum

$$H = \widehat{\bigoplus_n} H_n$$

is the Hilbert space obtained by completing $\bigoplus_n H_n$ with respect to the hermitian product

$$\langle (x_n)_n, (y_n)_n \rangle := \sum_n \langle x_n, y_n \rangle.$$

Concretely, $H$ is the space of sequences $(x_n)_n$ with $x_n \in H_n$ and $\sum_n \|x_n\|^2 < \infty$ (with the hermitian product above).

# Operators on Hilbert spaces I

## Definition

An *operator* on an Hilbert space $H$ is a continuous linear map $T\colon H \to H$.

## Definition

A linear operator $T\colon H \to H$ is called *bounded* if its domain is all $H$ and if there exists a constant $c$ such that $\|T(v)\| \leq c\|x\|$ for $x \in H$. In this case the smallest $c$ is called the *operator norm* of $T$ and is denoted $\|T\|$.

## Definition

Any operator $T$ has an *adjoint operator* $T^*$, characterized by

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for } x, y \in H.$$

A bounded operator is called *self-adjoint* (or *symmetric*) if $T^* = T$.

# Operators on Hilbert spaces II

**Definition**

We call $x$ an *eigenvector* of $T$ with *eigenvalue* $\lambda$ if $x \neq 0$ and $T(x) = \lambda x$. Given $\lambda$, the set of eigenvectors with eigenvalue $\lambda$ is called the $\lambda$-*eigenspace*.

**Remark**

If $T$ is a self-adjoint bounded operator, then its eigenvalues are real and the eigenspaces corresponding to distinct eigenvalues are orthogonal.

**Definition**

An operator $T\colon H \to H$ is called *compact* if whenever $(x_i)$ is a bounded sequence in $H$ then its image $T(x_i)$ has a convergent subsequence in $H$.

# Operators on Hilbert spaces III

## Properties of self-adjoint operators

- Let $T$ be a self-adjoint operator on $H$. Then

$$\|T\| = \sup_{\|x\|=1} |\langle T(x), x\rangle|,$$

where $\|T\|$ is the operator norm of $T$.
- Either $\|T\|$ or $-\|T\|$, or possibly both, are eigenvalues of $T$.

## Theorem 1 [Spectral theorem for compact self-adjoint operators]

Let $T$ be a compact self-adjoint operator on a separable Hilbert space $H$. Then $H$ has an orthonormal basis $e_i$ $(i = 1, 2, 3, \ldots)$ of eigenvectors of $T$, so that $T(e_i) = \lambda_i e_i$. The eigenvalues $\lambda_i \to 0$ as $i \to \infty$.

# Hilbert-Schmidt operators I

**Definition**

An operator $T : H \to H$ is called *Hilbert-Schmidt* if $H$ has an orthonormal basis $(e_i)_i$ such that

$$\sum_i \| T(e_i)\|^2 < \infty. \tag{1}$$

**Proposition 1**

Every Hilbert-Schmidt operator is compact.

# Hilbert-Schmidt operators II

**Proof**

Let $T\colon H \to H$ be a Hilbert-Schmidt operator, and let $(e_i)_i$ be an orthonormal basis of $H$ satisfying (1). For $n \geq 1$, define the $n$-truncation $T_n$ of $T$ as follows:

$$T_n(e_i) := \begin{cases} T(e_i) & \text{if } 1 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $x = \sum_{i=1}^{\infty} a_i e_i \in H$ such that $\|x\| = 1$. We have

$$T_n(x) = T_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i T(e_i)$$

and $T_n$ is a finite rank operator because its image is spanned by the finite set of vectors $T(e_1)$, …, $T(e_n)$.

# Hilbert-Schmidt operators III

We claim that $\|T - T_n\| \to 0$. Indeed by linearity and definition of $T_n$:

$$(T - T_n)(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=n+1}^{\infty} a_i\, T(e_i).$$

Thus

$$\|(T - T_n)(\sum_{i=1}^{\infty} a_i e_i)\| = \|\sum_{i=n+1}^{\infty} a_i\, T(e_i)\| \leq \sum_{i=n+1}^{\infty} |a_i|\, \|T(e_i)\|$$

$$\leq \left(\sum_{i=n+1}^{\infty} |a_i|^2\right)^{1/2} \cdot \left(\sum_{i=n+1}^{\infty} \|T(e_i)\|^2\right)^{1/2}$$

$$\leq \|\sum_{i=1}^{\infty} a_i e_i\| \cdot \left(\sum_{i=n+1}^{\infty} \|T(e_i)\|^2\right)^{1/2} \leq \left(\sum_{i=1}^{\infty} \|T(e_i)\|^2\right)^{1/2} < +\infty.$$

Thus $\|T - T_n\| \leq \left(\sum_{i=1}^{\infty} \|T(e_i)\|^2\right)^{1/2} < +\infty$. Hence $\|T - T_n\| \to 0$, and $T$ is compact as a limit of compact operators. □

# A key example of Hilbert-Schmidt operator I

For $(X, \mu)$ a measure space, let $L^2(X, \mu)$ denote the space of all complex-valued functions on $X$ which are square summable with respect to $\mu_X$.

**Theorem 2**

If $H = L^2(X, \mu)$ is separable, and if $K \in L^2(X \times X, \mu \times \mu)$ then the operator defined by

$$T_K(f)(x) := \int_X K(x, y) f(y) d\mu(y) \tag{2}$$

is Hilbert-Schmidt.

# Representations I

**Definition**

A *representation* $(\pi, V)$ of a group or an associative algebra $A$ (also called a left $A$-module) is a vector space $V$ equipped with a homomorphism $\pi \colon A \to \mathrm{End}(V)$, i.e., a linear map preserving the multiplication and unit.

If $V_1$, $V_2$ are two representations of $A$ then the direct sum $V_1 \oplus V_2$ has an obvious structure of a representation of $A$.

**Definition**

- A *subrepresentation* of a representation $(\pi, V)$ is a subspace $U \subset V$ which is invariant under all operators $\pi(a)$, $a \in A$.
- A nonzero representation $(\pi, V)$ is said to be *irreducible* if its only subrepresentations are 0 and $V$ itself, and *indecomposable* if it cannot be written as a direct sum of two nonzero subrepresentations.

# Representations II

## Definition

- Let $(\pi_1, V_1)$, $(\pi_2, V_2)$ be two representations of an algebra $A$. A *homomorphism* (or *intertwining operator*) $\phi\colon V_1 \to V_2$ is a linear operator which commutes with the action of $A$, i.e.,

$$\phi(\pi_1(a)v) = \pi_2(a)\phi(v) \quad \text{for any } v \in V_1.$$

- A homomorphism $\phi$ is said to be an *isomorphism of representations* if it is an isomorphism of vector spaces.
- The set (space) of all homomorphisms of representations $V_1 \to V_2$ is denoted by $\mathrm{Hom}_A(V_1, V_2)$.
- Two representations between which there exists an isomorphism are said to be isomorphic.

## Schur's lemma

Let $V_1$, $V_2$ be representations of an algebra $A$ over any field $C$ (which need not be algebraically closed). Let $\phi\colon V_1 \to V_2$ be a nonzero homomorphism of representations. Then:

1. If $V_1$ is irreducible, $\phi$ is injective;
2. If $V_2$ is irreducible, $\phi$ is surjective.

Thus, if both $V_1$ and $V_2$ are irreducible, then $\phi$ is an isomorphism.

## Proof

1. The kernel $U_1$ of $\phi$ is a subrepresentation of $V_1$. Since $\phi \neq 0$, this subrepresentation cannot be $V_1$. So by irreducibility of $V_1$ we have $U_1 = 0$.
2. The image $U_2$ of $\phi$ is a subrepresentation of $V_2$. Since $\phi \neq 0$, this subrepresentation cannot be 0. So by irreducibility of $V_2$ we have $U_2 = V_2$.

□

### Schur's lemma for algebraically closed fields

Let $V$ be a finite dimensional irreducible representation of an algebra $A$ over an algebraically closed field $k$, and $\phi\colon V \to V$ is an intertwining operator. Then $\phi = \lambda \mathrm{I}$ for some $\lambda \in k$ (a scalar operator).

### Proof

Let $\lambda$ be an eigenvalue of $\phi$ (a root of the characteristic polynomial of $\phi$). It exists since $k$ is an algebraically closed field. Then the operator $\phi - \lambda \mathrm{I}$ is an intertwining operator $V \to V$, which is not an isomorphism (since its determinant is zero). Thus by Schur's lemma this operator is zero, hence the result. □

One major goal of the theory of unitary representations is to decompose a unitary representation into simpler pieces. To understand the decomposition of representations into smaller pieces, we also need infinite "direct sums" of representations, hence the concept of a direct sum of Hilbert spaces.

## Defition

A *unitary representation* of a group $G$ is a homomorphism

$$\pi\colon G \to \mathrm{U}(H)$$

to the unitary group $\mathrm{U}(H) := \{g \in \mathrm{GL}(H) \,:\, g^* = g^{-1}\}$ of a complex Hilbert space $H$.

## Definition

A unitary representation $(\pi, H)$ of $G$ has a discrete decomposition if there are irreducible unitary subrepresentations $(\pi_n, H_n)$ such that $H = \widehat{\bigoplus}_n H_n$, and each occurs with finite multiplicity.

If $G$ is a locally compact group, then $G$ carries both a left and a right Haar measure. These are positive regular measures which are invariant under left and right translation, respectively, and are unique up to positive scalar multiples. The group $G$ is called *unimodular* if the left Haar measure is also a right Haar measure.

## Definition

Let $G$ be a locally compact (and countable at infinity), unimodular group (with Haar measure denoted $\mu$). A *cocompact lattice* in $G$ is a discrete subgroup $\Gamma$ such that $\Gamma\backslash G$ is compact.

Let $\Gamma$ be a cocompact lattice in $G$. Then $\mu$ induces a measure $\mu_X$ on $X := G/\Gamma$ which is left $G$-invariant.

We define the *leftt quasi-regular representation* of $G$, in the space $H := L^2(X, \mu)$, by

$$\lambda_g f(x) := f(g^{-1}x), \quad x \in X,\ g \in G,\ f \in L^2(X, \mu). \tag{3}$$

## Notation

For $f \in \mathcal{C}_c(G)$, let $T_f$ be the operator

$$T_f \colon h \mapsto f \cdot h \quad \text{where} \quad f \cdot h \colon x \mapsto \int_G f(g)h(g^{-1}x)dg. \tag{4}$$

Let $K_f(x,y) := \int_\Gamma f(x\gamma y^{-1})d\gamma$. We have

$$T_f(h)(x) = \int_G f(xg'^{-1})h(g')dg' = \int_X K_f(x,y)h(y)dy.$$

## Theorem 2

If $(\pi, H)$ is a unitary representation of $G$ such that $T_f$ is a compact operator on $H$ for all $f \in \mathcal{C}_c(G)$, then $H$ has a discrete decomposition.

## Proof (sketch)

- The main step is to show that any nonzero subrepresentation $(\pi', V')$ of $(\pi, H)$ contains an irreducible subrepresentation. For this, we pick $f \in \mathcal{C}_c(G)$ such that $T := T_f|_{V'}$ is nonzero and self-adjoint. As $T$ is also compact, it has a non-zero eigenvalue $\lambda$. Among stable subspaces $W$ of $V'$ for which $W[\lambda] := \ker(T - \lambda) \neq \{0\}$, we pick one that minimises $\dim W[\lambda]$, and pick $w_1 \in W[\lambda]$ such that $w_1 \neq 0$.

- We claim that $W_1 := \mathrm{Span}(G \cdot w_1)$ is irreducible. If not, we have $W_1 = U_1 \oplus U_2$, orthogonal sum of non-zero subrepresentations. Then $U_1$, $U_2$ are stable under $T$ and $W_1[\lambda] = U_1[\lambda] \oplus U_2[\lambda]$. By minimality of $W$ one of the $U_i[\lambda]$'s is $\{0\}$, say $U_2[\lambda]$. Hence $w_1 \in U_1$, but then $W_1 \subset U_1$ and $U_2 = \{0\}$, a contradiction.
- Next we show that $H$ is a Hilbert direct sum of irreducible subrepresentations. A set of irreducible and pairwise orthogonal subrepresentations of $H$ is called an orthogonal family. One checks that there is a maximal orthogonal family $A$.
- The orthogonal $Y$ of $\sum_{\tau \in A} \tau$ (equivalently of $\widehat{\bigoplus}_{\tau \in A} \tau$) is a subrepresentation containing no irreducible subrepresentation (by maximality of $A$), thus by the first step $Y = 0$ and $H = \widehat{\bigoplus}_{\tau \in A} \tau$.
- Finally, we check that multiplicities are finite. Say $\tau_1, \ldots, \tau_n$ are irreducible, pairwise isomorphic, and all appear in $H$. Pick $f \in \mathcal{C}_c(G)$ such that $T_f$ is self-adjoint and nonzero on $\tau_1$, and pick a nonzero eigenvalue $\lambda$ of $T_f$ on $\tau_1$. The eigenspaces $\tau_i[\lambda]$ are all isomorphic to $\tau_1[\lambda]$ (as $\tau_i \simeq \tau_1$), in particular nonzero, and are in direct sum inside $H[\lambda]$, thus $n \leq \dim H[\lambda] < \infty$ (as $T_f$ is compact). □

## Theorem 3

Let $G$ be a unimodular, locally compact group and let $\Gamma$ be a unimodular closed subgroup (e.g. a lattice) such that $X := G/\Gamma$ is compact.
Then $L^2(X)$ with the natural unitary action of $G$ (by left translation) has a discrete decomposition.

## Proof

Indeed, we have already seen that the operator $T_f$ is Hilbert-Schmidt on $L^2(X)$, thus it is compact by Proposition 1, so Theorem 2 applies. □