Local representation theory and Hecke algebras II

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Smooth representations of p-adic groups I

Notation

- F: non-archimedean local field (i.e. a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((T))$)
- G: F-rational points of a connected reductive algebraic group **G** defined over F (examples: $\mathrm{SL}_n(F)$, $\mathrm{GL}_n(F)$, $\mathrm{Sp}_{2n}(F)$)
- (π, V): representation of G.

Definition

A vector $v \in V$ is called *smooth*, if the isotropy group of v is an open subgroup of G.

The set V_∞ of smooth vectors forms a vector subspace of V, which is stable under G. In this way, one gets a G-module (π_∞, V_∞).

Smooth representations of p-adic groups II

Let (π, V) be a representation of G. The dual space V^* of V is equipped with the representation π^* of G defined by

$$\pi^*(g) \cdot v^*(v) := v^*(\pi(g^{-1}) \cdot v), \quad \text{for } v^* \in V^* \text{ and } v \in V. \qquad (1)$$

Matrix coefficients of a representation

For every $v \in V$ and $v^* \in V^*$, the matrix coefficient of (π, V) is the function $c_{v,v^*} : G \to \mathbb{C}$ defined by

$$c_{v,v^*}(g) := \langle v, \pi^*(g) v^* \rangle.$$

Contragredient representation

If (π, V) is smooth, the representation (π^*, V^*) is not necessarily smooth. Let $\widetilde{V}$ denote the smooth part of V^*, it is stable under the action of G. We denote by $(\widetilde{\pi}, \widetilde{V})$ the restriction of π^* to $\widetilde{V}$: it is a smooth representation of G, called the *contragredient representation* of (π, V).

Smooth representations of p-adic groups III

Remark

Smoothness is preserved by surjective morphisms and by the operation of taking subrepresentations, subquotients, direct sums.

Induction

Let H be a closed subgroup of G and let (τ, V_τ) be a smooth representation of H. Let $\mathrm{Ind}_H^G(V_\tau)$ denote the space of functions $f\colon G \to V_\tau$ such that

(1) for all $h \in H$ and $g \in G$, we have $f(hg) = \tau(h)f(g)$;

(2) there exists an open subgroup K_f of G such that for all $k \in K_f$ and $g \in G$, we have $f(gk) = f(g)$.

The space $\mathrm{Ind}_H^G(V_\tau)$ is stable under right translations by elements of G and condition (2) is the required smoothness condition, hence the representation $(\mathrm{Ind}_H^G\tau, \mathrm{Ind}_H^G V_\tau)$ of G is smooth and called the representation (smoothly) induced by τ.

Compact induction

The subspace $\mathrm{c-Ind}_H^G(V_\tau)$ of $\mathrm{Ind}_H^G(V_\tau)$ of functions with compact support modulo H (that is, the support is contained in some $H\Omega$ where Ω is a compact set in G) is G-stable and provides a subrepresentation $(\mathrm{c-Ind}_H^G(\tau), \mathrm{c-Ind}_H^G(V_\tau))$ of G called the representation *compactly induced* by τ.

Remark

The two representations $\mathrm{c-Ind}_H^G(V_\tau)$ and $\mathrm{Ind}_H^G(V_\tau)$ coincide when the quotient space G/H is compact.

Let $\mathfrak{R}(G)$ denote the category of all smooth complex representations of G. This is an abelian category admitting arbitrary coproducts.

Smooth representations of p-adic groups V

Definition

- An F-subgroup $\mathbf{P}$ of $\mathbf{G}$ is called a *parabolic subgroup* if the quotient variety $\mathbf{G}/\mathbf{P}$ is complete.
- If $\mathbf{B}$ is a parabolic subgroup of $\mathbf{G}$ which is solvable, then $\mathbf{B}$ is called a *Borel subgroup* of $\mathbf{G}$.

Levi decomposition

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$.

- The set of all closed connected normal unipotent subgroups of $\mathbf{P}$ has a unique maximal element $\mathbf{U}$ called the *unipotent radical* of P.
- There exists a reductive subgroup $\mathbf{L}$ of $\mathbf{G}$, called a *Levi factor* of $\mathbf{P}$, such that $\mathbf{L}$ normalizes $\mathbf{U}$ and $\mathbf{P} = \mathbf{L} \ltimes \mathbf{U}$.

Smooth representations of p-adic groups VI

- Let $\mathbf{P}$ be an F-parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$.
- Let P denote the group of F-rational points of $\mathbf{P}$.
- The unipotent radical $\mathbf{U}$ of $\mathbf{P}$ is F-rational. The group U of F-rational points of $\mathbf{U}$ is called the unipotent radical of P.
- The group P decomposes as a semidirect product $P = L \ltimes U$, where the group L (called a Levi factor of P) is the group of F-rational points of $\mathbf{L}$.
- We have a canonical isomorphism $L \simeq P/U$ obtained by composing the injection $L \hookrightarrow P$ with quotient map $P \to P/U$.
- Then P is a closed subgroup of G with a compact quotient G/P and we can induce representations from P to G. In this case there is no difference between smooth induction and compact induction.

Parabolic induction

Let $P = LU \supset B$ be a parabolic subgroup of G. Let $\overline{U}$ be the unipotent radical opposite to U and write $\overline{P} := L\overline{U}$. Let $\mathbf{P}$ be an F-parabolic subgroup of $\mathbf{G}$. We define the *modulus character* δ_P of P by

$$\int_P f(p^{-1}p'p)\mu_P(p') = \delta_P(p)\int_P f(p')\mu_P(p'), \quad \text{for any } p \in P, \qquad (2)$$

where μ_P is a Haar measure on P.

If σ is any representation of L, we may extend it in a trivial way to a representation $\sigma_P := \inf_L^P(\sigma)$ of P by setting

$$\sigma_P(lu) := \sigma(t) \quad \text{for any } l \in T \text{ and } u \in U.$$

The normality of U implies that σ_P is well-defined.

Definition

The parabolic induction functor is the following composition

$$\mathrm{i}_{L,P}^G : \mathfrak{R}(L) \xrightarrow{\mathrm{inf}_L^P} \mathfrak{R}(P) \xrightarrow{\mathrm{Ind}_P^G} \mathfrak{R}(G). \qquad (3)$$

Parabolic restriction

Since U is normal in P , for any smooth representation (τ, V_τ) of P the space

$$V(U) := \langle \pi(u)v - v \,:\, v \in V, u \in U \rangle$$

is stable under P and $V_U := V/V(U)$ provides a smooth representation of P that is trivial on U, hence a smooth representation of L. The parabolic restriction functor, or Jacquet (restriction) functor, is the following composition

$$\mathrm{r}_{L,P}^G : \mathfrak{R}(G) \xrightarrow{\mathrm{Res}_P^G} \mathfrak{R}(P) \xrightarrow{U\text{-coinvariants}} \mathfrak{R}(L). \qquad (4)$$

Proposition

Let $P = LU$ be a parabolic subgroup of G. Then the functors $\mathrm{i}_{L,P}^G$ and $\mathrm{r}_{L,P}^G$ are both exact, and $\mathrm{r}_{L,P}^G$ is the left adjoint of $\mathrm{i}_{L,P}^G$.

Theorem [Bernstein]

The right adjoint of $\mathrm{i}_{L,P}^G$ is the functor $\mathrm{r}_{L,\overline{P}}^G$, where $\overline{P}$ denotes the opposite parabolic subgroup of P.

Definition

The set of all irreducible representations of G which are subquotients of $\mathrm{i}_{T,B}^G(\chi)$, with B a Borel subgroup of G and χ of character of a torus $T \subset B$ is called the *principal series* of representations of G.

Smooth representations of p-adic groups X

Definition

A smooth representation π of G is *supercuspidal* if $\mathrm{r}^G_{L,P}(\pi) = 0$, for any proper parabolic subgroup P of G.

Proposition

Let π be a smooth representation of G. The following conditions are equivalent

(1) the representation π is supercuspidal;

(2) the representation π not a subquotient of any proper parabolically induced representation.

(3) every matrix coefficient of π has compact modulo center support.

Smooth representations of p-adic groups XI

Theorem [Harish-Chandra]

Any smooth irreducible π of G occurs as an irreducible component of a parabolically induced representation $\mathrm{i}_{L,P}^{G}(\sigma)$, where P is a parabolic subgroup of G with Levi factor L and $\sigma \in \mathrm{Irr}(L)$ is supercuspidal.
The G-conjugacy class $(L,\sigma)_G$ of (L,σ) is uniquely determined and is called the supercuspidal support of π, it is denoted by $\mathrm{Sc}(\pi)$.

Definition

Let L be a Levi subgroup of a parabolic subgroup P of G. A character $\chi\colon L \to \mathbb{C}^\times$ is *unramified* if χ is trivial on every compact subgroup of L.
Let $\mathfrak{X}_{\mathrm{nr}}(L)$ denote the group of unramified characters of L.

Let σ be an irreducible supercuspidal smooth representation of L and $\mathcal{O}$ the set of equivalence classes of representations L of the form $\sigma \otimes \chi$, with $\chi \in X_{\mathrm{nr}}(L)$. We write $\mathfrak{s} := (L,\mathcal{O})_G$ for the G-conjugacy class of the pair $(L,\mathcal{O})$ and $\mathfrak{B}(G)$ for the set of such classes $\mathfrak{s}$. We set $\mathfrak{s}_L := (L,\mathcal{O})_L$.

Smooth representations of p-adic groups XII

We denote by $\mathfrak{R}^{\mathfrak{s}}(G)$ the full subcategory of $\mathfrak{R}(G)$ whose objects are the representations (π, V) such that every G-subquotient of π is equivalent to a subquotient of a parabolically induced representation $\mathrm{i}_{L,P}^{G}(\sigma')$, where $\sigma' \in \mathcal{O}$.

Theorem [Bernstein]

The categories $\mathfrak{R}^{\mathfrak{s}}(G)$ are indecomposable and split the full smooth category $\mathfrak{R}(G)$ in a direct product:

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}^{\mathfrak{s}}(G). \tag{5}$$

Let $\mathrm{Irr}^{\mathfrak{s}}(G)$ denote the set of irreducible objects of the categoty $\mathfrak{R}^{\mathfrak{s}}(G)$. As a direct consequence of the decomposition above, we have

$$\mathrm{Irr}(G) = \coprod_{\mathfrak{s}\in\mathfrak{B}(G)} \mathrm{Irr}^{\mathfrak{s}}(G). \tag{6}$$

Let $\mathfrak{s} = [L, \sigma]_G \in \mathfrak{B}(G)$. We denote by L^1 the intersection of kernels of unramified characters of L. Let (σ_1, V_1) be an irreducible component of the restriction of σ to L^1. The isomorphism class of

$$\Pi^{\mathfrak{s}_L} := \mathrm{c}-\mathrm{Ind}_{L^1}^{L}(\sigma_1, V_1) \tag{7}$$

is independent of the choice of (σ_1, V_1).

Theorem [Bernstein]

The functor $V \mapsto \mathrm{Hom}_G(\Pi^{\mathfrak{s}}, V)$ is an equivalence from $\mathfrak{R}^{\mathfrak{s}}(G)$ to the algebra $\mathcal{H}^{\mathfrak{s}}(G) := \mathrm{End}_G(\Pi^{\mathfrak{s}})$.

Theory of types

We fix a Haar measure on G. Let $\mathcal{H}(G)$ be the set of complex-valued, locally constant, compactly supported functions f on G. This is an algebra with respect to convolution

$$f_1 * f_2(g) := \int_G f_1(x) f_2(x^{-1}g)\,dx. \tag{8}$$

Smooth representations of p-adic groups XV

Let (λ, V_λ) be a smooth representation of a compact open subgroup J of G, and let $(\tilde{\lambda}, V_{\tilde{\lambda}})$ denote its contragredient. We define $\mathcal{H}(G,\lambda)$ to be the space of compactly supported functions $f\colon G \to \mathrm{End}_G(V_{\tilde{\lambda}})$ such that

$$f(jgj') = \tilde{\lambda}(j)f(g)\tilde{\lambda}(k'), \quad \text{where } j, j' \in K \text{ and } g \in G. \tag{9}$$

The convolution product gives $\mathcal{H}(G,\lambda)$ the structure of a unitary associative $\mathbb{C}$-algebra.

Let $e_\lambda \in \mathcal{H}(G)$ be the function defined by

$$e_\lambda(g) := \begin{cases} \frac{\dim \lambda}{\mu(J)} \mathrm{tr}(\lambda(g^{-1})) & \text{if } g \in J, \\ 0 & \text{if } g \in G,\ g \notin J. \end{cases}$$

Then e_λ is idempotent, and $e_\lambda \star \mathcal{H}(G) \star e_\lambda$ is a sub-algebra of $\mathcal{H}(G)$ with unit e_λ.

Smooth representations of p-adic groups XVI

Bushnell and Kutzko defined a canonical isomorphism:

$$\mathcal{H}(G,\lambda) \otimes_{\mathbb{C}} \mathrm{End}_{\mathbb{C}}(V_\lambda) \to e_\lambda \star \mathcal{H}(G) \star e_\lambda.$$

The algebras $\mathcal{H}(G,\lambda)$ and $e_\lambda \star \mathcal{H}(G) \star e_\lambda$ are therefore canonically Morita equivalent. Hence, we get an equivalence of categories:

$$\mathcal{H}(G,\lambda) - \mathrm{Mod} \approx e_\lambda \star \mathcal{H}(G) \star e_\lambda - \mathrm{Mod}. \quad (10)$$

We write $\mathfrak{R}_\lambda(G)$ for the full sub-category of $\mathfrak{R}(G)$ whose objects are those V satisfying $V = \mathcal{H}(G) \star e_\lambda \star V$, that is, $\mathfrak{R}_\lambda(G)$ is generated over G by the subspace $e_\lambda \star V$.

Definition

Let $\mathfrak{S}$ be a finite subset of $\mathfrak{B}(G)$. A pair (J, λ) is an $\mathfrak{S}$-*type* for G if the following property is satisfied

$$\lambda \text{ occurs in the restriction of } \pi \in \mathrm{Irr}(G) \text{ if and only if } \pi \in \mathfrak{S}. \tag{11}$$

If $\mathfrak{S}$ has only one element $\mathfrak{s}$, we then refer to an $\mathfrak{s}$-type, rather than an $\{\mathfrak{s}\}$-type.

Example

The pair $(I, 1)$, where I is an Iwahori subgroup of G and 1 the trivial representation of I, is an $\mathfrak{s}$-type for $\mathfrak{s} = [T, 1]_G$, where T is a the group of F-points of a split torus and 1 the trivial representation of T.

Smooth representations of p-adic groups XVIII

Theorem [Bushnell-Kutzko]

Let $\mathfrak{S}$ be a finite subset of $\mathfrak{B}(G)$. If (J, λ) is an $\mathfrak{S}$-type for G, then

$$\mathfrak{R}_\lambda(G) = \prod_{\mathfrak{s} \in \mathfrak{S}} \mathfrak{R}^{\mathfrak{s}}(G), \tag{12}$$

as a product of subcategories.
In particular, if (J, λ) is an $\mathfrak{s}$-type for G, then we have $\mathfrak{R}_\lambda(G) = \mathfrak{R}^{\mathfrak{s}}(G)$, and the latter is equivalent to the category of modules of $\mathcal{H}(G, \lambda)$.

Consequence

In order to understand the category $\mathfrak{R}(G)$, we are reduced to study the categories of modules of the algebras $\mathcal{H}^{\mathfrak{s}}(G)$ (point of view of the spherical building of G) or $\mathcal{H}(G, \lambda)$ (point of view of the affine building of G). So we need to understand the structure of these algebras.

Unramified representations I

Definition

Let K be a compact open subgroup of G. We denote by $\mathcal{H}(G, K)$ the unital algebra $e_K \mathcal{H}(G) e_K$, where $e_K := \mathrm{vol}(K)^{-1} \mathbf{1}_K$, consisting of compactly supported bi-K-invariant locally constant functions on G.

Proposition

Let $G = \mathrm{GL}_n(F)$ and $K_0 := \mathrm{GL}_n(\mathfrak{o}_F)$, where $\mathfrak{o}_F$ is the ring of integers of F. The *spherical Hecke algebra* $\mathcal{H}(G, K_0)$ is commutative.

Definition

A smooth representation π of G is said to be *spherical* (or *unramified*) if $\pi^{K_0} \neq 0$.

Unramified representations II

Corollary

Irreducible unramified representations of $G = \mathrm{GL}_n(F)$ correspond bijectively with algebra homomorphisms $\mathrm{Hom}_{\mathbb{C}\text{–alg}}(\mathcal{H}(G, K_0), \mathbb{C})$.

Proof

We use the fact that, for any open compact subgroup K, the map $V \mapsto V^K$ is a bijection between irreducible smooth representations (π, V) such that $\pi^K \neq 0$ and simple $\mathcal{H}(G, K)$-modules. Since $\mathcal{H}(G, K_0)$ is commutative, its simple modules are one-dimensional.

Definition

An irreducible representation (π, V) of G is said to have *depth zero* if $V^{G_{x,0+}} \neq \{0\}$ for some point x of the Bruhat-Tits building of G, where $G_{x,0+}$ is the pro-p unipotent radical of the parahoric subgroup $G_{x,0}$.

Unramified representations III

Remark

Every irreducible Iwahori-spherical representation has depth zero.

Theorem [Morris, Moy-Prasad]

Let π be an irreducible depth-zero supercuspidal representation of G. Then there exists a maximal parahoric subgroup $G_{x,0}$ such that π is compactly induced from a representation ρ of the normalizer of $G_{x,0}$ in G such that the restriction of ρ to $G_{x,0+}$ is 1-isotypic.

Example

Let τ be an irreducible cuspidal representation of $\mathrm{GL}_N(\mathbb{F}_q)$, let $\widetilde{\tau}$ be the inflation of τ to the maximal compact subgroup K_0 of $\mathrm{GL}_n(F)$. We have $K_0 = G_{x,0}$, where x is a vertex in the Bruhat-Tits building of $\mathrm{GL}_n(F)$. The normalizer of K_0 in G is the group $F^\times K_0$. Let τ denote the extension to $F^\times K_0$ of $\boldsymbol{\tau}$. The representation $\mathrm{c-Ind}_{F^\times K_0}^{\mathrm{GL}_n(F)} \boldsymbol{\tau}$ is an irreducible depth-zero supercuspidal representation of $\mathrm{GL}_n(F)$.

Hecke algebra of a Coxeter system I

Definition

A *Coxeter group* W is a group generated by a set S of elements of order 2, which has a presentation

$$W = \langle S \ : \ (ss')^{m(s,s')} = 1 \text{ for all } s, s' \in S \rangle,$$

where $m(s,s') \in \mathbb{Z}_{\geq 1}$ is the order of ss' in W.

The equalities $s^2 = 1$ (for $s \in S$) are called the *quadratic relations*, while the equalities $(ss')^{m(s,s')} = 1$, or equivalently

$$\underbrace{ss'ss'\cdots}_{m(s,s') \text{ terms}} = \underbrace{s'ss's\cdots}_{m(s,s') \text{ terms}} \tag{13}$$

are called the *braid relations*.

Hecke algebra of a Coxeter system II

Examples of finite Coxeter groups

1. $W = S_n$, with $S = \{(12), (23), \ldots, (n-1n)\}$, symmetric group, type A_{n-1};
2. $W = S_n \ltimes \{\pm 1\}^n$, with $S = \{(12), (23), \ldots, (n-1n), (\mathrm{id}, (1, \ldots, 1, -1))\}$, hyperoctahedral group, type B_n or C_n.

Let (W, S) be a Coxeter group. Given $w \in W$ we denote by $\ell(w)$ its word length with respect to the generating set S. A word w with letters in S is called *reduced* if it has $\ell(w)$ many letters.

Hecke algebra of a Coxeter system III

Proposition

Let (W, S) be a Coxeter group, equipped with a function $q\colon s \mapsto q_s$ from S to $\mathbb{C}$ satisfying

$$q_s = q_{s'} \quad \text{if } s \text{ and } s' \text{ are conjugate under } W. \tag{14}$$

There is a unique algebra structure $\mathcal{H}(W, q)$ on the $\mathbb{C}$-vector space spanned by elements T_w ($w \in W$) such that

(1) $T_1 = 1$,

(2) $(T_s - q_s)(T_s + 1) = 0$ for any $s \in S$ (quadratic relations),

(3) $T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots$ where both sides have $m_{s,s'}$ elements (braid relations),

(3) $T_{ww'} = T_w T_{w'}$ if $\ell(ww') = \ell(w) + \ell(w')$.

The algebra $\mathcal{H}(W, q)$ is called the Hecke-Iwahori algebra of (W, q).

Iwahori-spherical representations I

Bruhat-Iwahori decomposition of G

Suppose that G is F-split. Let $W := \mathrm{N}_G(T)/\mathbf{T}(\mathfrak{o}_F)$. Assume for today that W is a Coxeter group. We have

$$W \longrightarrow I\backslash G/I \tag{15}$$

assigning to $w \in W$ its Bruhat cell IwI.

Notation

Let $\mathbf{1}_E$ denote the characteristic function of a subset E of G.

Iwahori-spherical representations II

Theorem

Each IgI is a finite union of cosets Ix, the algebra $\mathcal{H}(G, I)$ is closed under convolution products, and

$$\mathbf{1}_{IsI} * \mathbf{1}_{IwI} = \mathbf{1}_{IswI} \quad \text{for } \ell(sw) > \ell(w) \tag{16}$$

$$\mathbf{1}_{IsI} * \mathbf{1}_{IsI} = a_s \mathbf{1}_{IsI} + b_s \mathbf{1}_I, \quad \text{with } a_s := q_s - 1 \text{ and } b_s = q_s. \tag{17}$$

That is, these Iwahori-Hecke operators form a generic algebra with the indicated structure constants. Further, for a reduced expression $w = s_1 \cdots s_n$ (that is, with $n = \ell(w)$ and all $s_i \in S$), we have

$$q_w = q_{s_1} \cdots q_{s_n}.$$

Iwahori-spherical representations III

Proof

First prove that double cosets IwI are finite unions of cosets Ix at the same time that we study one of the requisite identities for the generic algebra structure. This also will prove that $\mathcal{H}(G, I)$ is closed under convolution products. Do induction on the length of $w \in W$. Take $s \in S$ so that $\ell(sw) > \ell(w)$. At $g \in G$ where $\mathbf{1}_{Is} * \mathbf{1}_{IwI}$ does not vanish, there is $h \in G$ so that

$$\mathbf{1}_{IsI}(gh^{-1})\mathbf{1}_{IwI}(h) \neq 0.$$

For such h, we have $gh^{-1} \in IsI$ and $h \in IwI$. Thus, by the Bruhat cell multiplication rules,

$$g = (gh^{-1})h \in IsI \cdot IwI = IswI.$$

Since this convolution product is left and right I-invariant

$$\mathbf{1}_{IsI} * \mathbf{1}_{IwI} = c \cdot \mathbf{1}_{IswI} \quad \text{for some positive rational number } c = c(s, w).$$

Iwahori-spherical representations IV

Compute the constant $c = c(s, w)$ by summing the previous equality over $I\backslash G$

$$\begin{aligned} c \cdot q_{sw} = c \cdot |I\backslash IswI| = c \cdot \sum_{g \in I\backslash G} \mathbf{1}_{IswI} = c \cdot \mathbf{1}_{IswI}(g) \\ = c \cdot \sum_{g \in I\backslash G} (\mathbf{1}_{IsI} * \mathbf{1}_{IwI})(g) = \sum_{g \in I\backslash G} \sum_{h \in I\backslash G} \mathbf{1}_{BsB}(gh^{-1}) \mathbf{1}_{IwI}(h) \\ = \sum_{g \in I\backslash G} \sum_{h \in I\backslash G} \mathbf{1}_{IsI}(g) \mathbf{1}_{IwI}(h) = q_s q_w \end{aligned} \tag{18}$$

(the latter by replacing g by gh, interchanging order of summation).

Iwahori-spherical representations V

Thus,

$$c = q_s q_w / q_{sw} \tag{19}$$

and for $\ell(sw) > \ell(w)$

$$\mathbf{1}_{IsI} * \mathbf{1}_{IwI} = q_s q_w q_{sw}^{-1} \mathbf{1}_{IswI}. \tag{20}$$

□

Remark

This shows that the cardinality q_{sw} of $I \backslash IwI$ is finite for all $w \in W$, and therefore that the Hecke algebra is closed under convolution.