Local representation theory and Hecke algebras III

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Unramified representations I

Definition

Let K be a compact open subgroup of G. We denote by $\mathcal{H}(G, K)$ the unital algebra $e_K\mathcal{H}(G)e_K$, where $e_K := \mathrm{vol}(K)^{-1}\mathbf{1}_K$, consisting of compactly supported bi-K-invariant locally constant functions on G.

Proposition

Let $G = \mathrm{GL}_n(F)$ and $K_0 := \mathrm{GL}_n(\mathfrak{o}_F)$, where $\mathfrak{o}_F$ is the ring of integers of F. The *spherical Hecke algebra* $\mathcal{H}(G, K_0)$ is commutative.

Definition

A smooth representation π of G is said to be *spherical* (or *unramified*) if $\pi^{K_0} \neq 0$.

Unramified representations II

Corollary

Irreducible unramified representations of $G = \mathrm{GL}_n(F)$ correspond bijectively with algebra homomorphisms $\mathrm{Hom}_{\mathbb{C}\text{–alg}}(\mathcal{H}(G, K_0), \mathbb{C})$.

Proof

We use the fact that, for any open compact subgroup K, the map $V \mapsto V^K$ is a bijection between irreducible smooth representations (π, V) such that $\pi^K \neq 0$ and simple $\mathcal{H}(G, K)$-modules. Since $\mathcal{H}(G, K_0)$ is commutative, its simple modules are one-dimensional.

Definition

An irreducible representation (π, V) of G is said to have *depth zero* if $V^{G_{x,0+}} \neq \{0\}$ for some point x of the Bruhat-Tits building of G, where $G_{x,0+}$ is the pro-p unipotent radical of the parahoric subgroup $G_{x,0}$.

Unramified representations III

Definition

A representation of G is called *Iwahori-spherical* if it has a non-zero fixed vector under some Iwahori subgroup of G.

Remark

Every irreducible Iwahori-spherical representation has depth zero.

Theorem [Morris, Moy-Prasad]

Let π be an irreducible depth-zero supercuspidal representation of G. Then there exists a maximal parahoric subgroup $G_{x,0}$ such that π is compactly induced from a representation ρ of the normalizer of $G_{x,0}$ in G such that the restriction of ρ to $G_{x,0+}$ is 1-isotypic.

Unramified representations IV

Example

Let τ be an irreducible cuspidal representation of $\mathrm{GL}_N(\mathbb{F}_q)$ (where $\mathbb{F}_q$ is the residual field of F), let $\mathrm{infl}(\tau)$ denote the inflation of τ to the maximal compact subgroup K_0 of $\mathrm{GL}_n(F)$.
We have $K_0 = G_{x,0}$, where x is a vertex in the Bruhat-Tits building of $\mathrm{GL}_n(F)$. The normalizer of K_0 in G is the group $F^\times K_0$.
Let $\boldsymbol{\tau}$ denote an extension of $\mathrm{infl}(\tau)$ to $F^\times K_0$.

- The representation $\mathrm{c}-\mathrm{Ind}_{F^\times K_0}^{\mathrm{GL}_n(F)}\boldsymbol{\tau}$ is an irreducible depth-zero supercuspidal representation of $\mathrm{GL}_n(F)$.
- Any irreducible depth-zero supercuspidal representation of $\mathrm{GL}_n(F)$ is of the form $\mathrm{c}-\mathrm{Ind}_{F^\times K_0}^{\mathrm{GL}_n(F)}\boldsymbol{\tau}$ for some irreducible cuspidal representation τ of $\mathrm{GL}_N(\mathbb{F}_q)$

Hecke algebra of a Coxeter system I

Definition

A *Coxeter group* W is a group generated by a set S of elements of order 2, which has a presentation

$$W = \langle S \;:\; (ss')^{m_{s,s'}} = 1 \text{ for all } s, s' \in S \rangle,$$

where $m_{s,s'} \in \mathbb{Z}_{\geq 1}$ is the order of ss' in W.

The equalities $s^2 = 1$ (for $s \in S$) are called the *quadratic relations*, while the equalities $(ss')^{m_{s,s'}} = 1$, or equivalently

$$\underbrace{ss'ss'\cdots}_{m_{s,s'} \text{ terms}} = \underbrace{s'ss's\cdots}_{m_{s,s'} \text{ terms}} \tag{1}$$

are called the *braid relations*.

Hecke algebra of a Coxeter system II

Examples of finite Coxeter groups

1. $W = S_n$, with $S = \{(12), (23), \ldots, (n-1n)\}$, symmetric group, type A_{n-1};
2. $W = S_n \ltimes \{\pm 1\}^n$, with $S = \{(12), (23), \ldots, (n-1n), (\mathrm{id}, (1, \ldots, 1, -1))\}$, hyperoctahedral group, type B_n or C_n.

Let (W, S) be a Coxeter group. Given $w \in W$ we denote by $\ell(w)$ its word length with respect to the generating set S. A word w with letters in S is called *reduced* if it has $\ell(w)$ many letters.

Hecke algebra of a Coxeter system III

Proposition

Let (W, S) be a Coxeter group, equipped with a function $q\colon s \mapsto q_s$ from S to $\mathbb{C}$ satisfying

$$q_s = q_{s'} \quad \text{if } s \text{ and } s' \text{ are conjugate under } W. \tag{2}$$

There is a unique algebra structure $\mathcal{H}(W, q)$ on the $\mathbb{C}$-vector space spanned by elements T_w $(w \in W)$ such that

(1) $T_1 = 1$,

(2) $(T_s - q_s)(T_s + 1) = 0$ for any $s \in S$ (quadratic relations),

(3) $T_s T_{s'} T_s \cdots = T_{s'} T_s T_{s'} \cdots$ where both sides have $m_{s,s'}$ elements (braid relations),

(3) $T_{ww'} = T_w T_{w'}$ if $\ell(ww') = \ell(w) + \ell(w')$.

The algebra $\mathcal{H}(W, q)$ is called the Hecke algebra of (W, q).

Recollection on buildings I

A *chamber complex* is a simplicial complex X such that all simplices of X are contained in a maximal simplex and such that for every pair x, y of maximal simplices of X there exists a chain $x = x_0$, x_1, ..., $x_n = y$ of adjacent maximal simplices in X. The maximal simplices are called *chambers*.

A chamber complex is called *thin* if each facet of a chamber is the facet of exactly two chambers, and it is called *thick* if each facet of a chamber is the facet of at least three chambers.

An important example of a thin chamber complex is the *Coxeter complex* $\Sigma(W, S)$ associated with a Coxeter system (W, S).

Recollection on buildings II

A *system of apartments* in a chamber complex X is a set $\mathcal{A}$ of thin chamber subcomplexes such that

- for each pair of simplices x, y of X there is $A \in \mathcal{A}$ such that $x, y \in A$, and
- if $A, A' \in \mathcal{A}$ contain a common chamber C and a simplex x, then there is an isomorphism of chamber complexes $A \to A'$ that fixes C and x point-wise.

Definition

A *thick building* is a thick chamber complex that admits a system of apartments. One can prove that for a thick building X there is a unique maximal apartment system. So if X is thick, as we assume from now on, it makes sense to speak of an apartment of X without reference to a specific system of apartments.

Recollection on buildings III

There is a Coxeter system (W, S) such that every apartment of X is chamber isomorphic with $\Sigma(W, S)$. The choice of a fundamental chamber C in X and a compatible labelling of its facets by elements of S makes these isomorphisms unique and introduces a labelling of all facets of chambers in X. The Coxeter system (W, S) is called the type of X.

Definition

Let X be a building of type (W, S). The *thickness* of X is the tuple $(d_s)_{s\in S}$ where $d_s - 1$ is the cardinality of the set of chambers containing the fundamental chamber's facet labelled by s (so that a building is thick if and only if $d_s \geq 2$ for all $s \in S$).
The building is called *locally finite* if all d_s, $s \in S$ are finite.

Iwahori-Hecke algebra I

Definition

Let I be an Iwahori subgroup of G. The algebra $\mathcal{H}(G, I)$ is defined to be the convolution algebra consisting of all compactly supported left and right I-invariant functions on G.

Theorem

Let X be the Bruhat-Tits building of G and denote by $(d_s)_{s\in S}$ the thickness of X. Let $q_s := d_s^{-1}$, for $s \in S$. Then, as $*$-algebras,

$$\mathcal{H}(G, I) \simeq \mathcal{H}(W, q) \tag{3}$$

with an isomorphism given by the map $T_w \mapsto (-1)^{\ell(w)}\sqrt{q}_w \mathbf{1}_{\mathrm{IwI}}$, where for an element $w \in W_{\mathrm{f}}$ with a reduced expression $w = s_1 s_2 \cdots s_r$

$$q_w := q_{s_1} q_{s_2} \cdots q_{s_r}. \tag{4}$$

Affine Hecke algebras I

Definition

A *root datum* is a 4-tuple $\mathcal{R} = (X, Y, R, R^\vee)$ such that X, Y are free abelian groups of finite rank together with a perfect pairing

$$\langle\ ,\ \rangle \colon X \times Y \to \mathbb{Z},$$

$R \subset X \backslash \{0\}$, $R^\vee \subset Y \backslash \{0\}$ are the finite sets of roots and coroots respectively, in bijection $\alpha \leftrightarrow \alpha^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$,

- for every $\alpha \in R$, the reflection

$$s_\alpha \colon X \to X, \quad s_\alpha(x) := x - \langle x, \alpha^\vee \rangle, \quad \text{stabilizes } R,$$

- for every $\alpha \in R^\vee$, the reflection

$$s_{\alpha^\vee} \colon Y \to Y, \quad s_{\alpha^\vee}(y) := y - \langle \alpha, y \rangle, \quad \text{stabilizes } R^\vee.$$

Affine Hecke algebras II

Notation

Often we will add a basis Δ to R, and speak of a based root datum.
The group $W_{\mathrm{f}} = W(R) \subset \mathrm{GL}(X)$ generated by the s_α with $\alpha \in \Delta$ is a finite Weyl group,

Definition

The infinite group

$$W_{\mathrm{ex}} := W_{\mathrm{f}} \rtimes X \tag{5}$$

is called the *extended affine Weyl group*.
The elements of W_{ex} are of the form wt_x, where $w \in W$ and $x \in X$ and we have:

$$(w_1 t_{x_1}) \cdot (w_2 t_{x_2}) = w_1 w_2 t_{w^{-1}(x_1)+x_2}.$$

The group $W_{\rm ex}$ acts naturally on the vector space $X \otimes_{\mathbb{Z}} \mathbb{R}$: $W_{\rm f}$ by linear extension of its action on X and X by translations. The collection of hyperplanes

$$H_{\alpha,n} := \{x \in X \otimes_{\mathbb{Z}} \mathbb{R} \,:\, \alpha^\vee(x) = n\} \quad \alpha \in R, n \in \mathbb{Z}, \}$$

is $W_{\rm ex}$-stable and divides $X \otimes_{\mathbb{Z}} \mathbb{R}$ in open subsets called *alcoves*.

Definition

Let $W_{\rm aff}$ be the subgroup of $W_{\rm ex}$ generated by the reflections in the hyperplanes $H_{\alpha,n}$. It is an affine Weyl group. The basis Δ determines a "fundamental alcove" A_0 in $X \otimes_{\mathbb{Z}} \mathbb{R}$, and let $S_{\rm aff}$ be the set of reflections with respect to the walls of A_0. Then $(W_{\rm aff}, S_{\rm aff})$ is a Coxeter system.

Affine Hecke algebras IV

We have

$$W_{\mathrm{ex}} = W_{\mathrm{aff}} \rtimes \Omega, \tag{6}$$

where Ω is the stabilizer of A_0 in W_{ex}.

The length function on $(W_{\mathrm{aff}}, S_{\mathrm{aff}})$ satisfies

$$\ell(w) = \text{number of hyperplanes } H_{\alpha,n} : \text{that separate } w(A_0) \text{ from } A_0. \tag{7}$$

We extend ℓ to a function $W(\mathcal{R}) \to \mathbb{Z}_{\geq 0}$, by decreeing that (7) is valid for all $w \in W_{\mathrm{ex}}$. Then

$$\Omega = \{w \in W(R) \,:\, \ell(w) = 0\}.$$

Example

Let $X := Y = \mathbb{Z}^n$,

$$R := R^\vee = \mathrm{A}_{n-1} = \{\varepsilon_i - \varepsilon_j \,:\, 1 \leq i, j \leq n, i \neq j\},$$

$\Delta := \{\varepsilon_i - \varepsilon_{i+1} \,:\, i = 1, \ldots, n-1\}$. Here S_{aff} is the union

$\{s_i = s_{\varepsilon_i - \varepsilon_{i+1}}\alpha \,:\, i = 1, \ldots, n{-}1\} \cup \{s_0 : x \mapsto x + (1 - \langle x, \varepsilon_1 - \varepsilon_n\rangle)(\varepsilon_1 - \varepsilon_n)\}$.

We have

$$W_{\mathrm{f}} \simeq S_n \quad \text{and} \quad W_{\mathrm{aff}} = \{(x_1, \ldots, x_n) \in \mathbb{Z}^n \,:\, x_1 + \cdots x_n = 0\} \rtimes S_n,$$

and Ω is isomorphic to $\mathbb{Z}$, generated by $\omega := \varepsilon_1(12 \cdots n)$. The action of ω on S_{aff} is $\omega s_i \omega^{-1} = s_{i+1}$ (where s_n means s_0).

Affine Hecke algebras VI

Notation

- Let $\mathcal{R} = (X, R, Y, R^\vee)$ be an irreducible root datum with finite Weyl group W_f,
- let $q \in \mathbb{R}_{\geq 1}$,
- let $\lambda, \lambda^* \colon R \to \mathbb{C}$ be W_f-invariant functions such that

$$\alpha \notin 2Y \;\Rightarrow\; \lambda(\alpha) = \lambda^*(\alpha). \tag{8}$$

- Let $\mathbb{C}[X]$ be the group algebra of X, with the standard basis $\{\theta_x, x \in X\}$ and let $\mathcal{H}(W_\mathrm{f}, q)$ be the Hecke algebra of W_f.

Affine Hecke algebras VII

Proposition

There is a unique algebra structure on the vector space

$$\mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) := \mathbb{C}[X] \otimes \mathcal{H}(W_{\mathrm{f}}, q)$$

such that the following relations are satisfied:

(1) $\mathbb{C}[X]$ and $\mathcal{H}(W_{\mathrm{f}}, q)$ are embedded as subalgebras,

(2) for $\alpha \in \Delta$ and $x \in X$:

$$\theta_x T_{s_\alpha} - T_{s_\alpha} \theta_{s_\alpha(x)} =$$

$$\Big((q^{\lambda(\alpha)} - 1) + \theta_{-\alpha}(q^{(\lambda(\alpha)+\lambda^*(\alpha))/2} - q^{(\lambda(\alpha)-\lambda^*(\alpha))/2})\Big) \frac{\theta_x - \theta_{s_\alpha(x)}}{\theta_0 - \theta_{-2\alpha}},$$

where $\{\theta_x \,:\, x \in X\}$ is a basis of $\mathbb{C}[X]$.

It is an associative algebra.

Affine Hecke algebras VIII

Theorem [Bernstein]

Suppose that $q_s \neq 0$ for all $s \in S_{\mathrm{aff}}$. Let $\lambda(\alpha)$, $\lambda^*(\alpha)$ such that

$$q_{s_\alpha} = q^{\lambda(\alpha)} \quad \text{for all } \alpha \in R \quad \text{and} \quad q_{s'_\alpha} = q^{\lambda^*(\alpha)} \quad \text{when } \alpha^\vee \in R^\vee_{\max},$$

where $R^\vee_{\max}$ is the set of maximal elements of $R^\vee$, with respect to $\Delta^\vee$. Then there exists a unique algebra isomorphism

$$\mathcal{H}(W_{\mathrm{aff}}, q) \longrightarrow \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q) \tag{9}$$

such that

- it is the identity on $\mathcal{H}(W_{\mathrm{f}}, q)$,
- for $x \in \mathbb{Z}R$ with $\langle x, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, it sends $q_x^{-1/2} T_x$ to θ_x.

Representations of affine Hecke algebras I

Notation

Let $\mathcal{R} = (X, R, Y, R^\vee, \Delta)$ be a based root datum and let J be a subset of Δ. Then $R_J := R \cap \mathbb{Q}J$ is a standard parabolic root subsystem of R, with dual root system $R_J^\vee := R^\vee \cap \mathbb{Q}J^\vee$, and $\mathcal{R}_J := (X, R_J, Y, R_J^\vee, J)$ is a the based root datum.

Let W_J denote the Weyl group of $\mathcal{R}_J$.

Any parameter functions λ, λ^* for $\mathcal{R}$ restrict to parameter functions for $\mathcal{R}_J$, and $\mathcal{H}_J := \mathcal{H}(\mathcal{R}_J, \lambda, \lambda^*, q)$ is a subalgebra of $\mathcal{H} := \mathcal{H}(\mathcal{R}, \lambda, \lambda^*, q)$.

The algebras $\mathcal{H}$ and $\mathcal{H}_J$ share the same commutative subalgebra $\mathbb{C}[X]$.

Definition

Parabolic induction for representations of affine Hecke algebras is defined to be the induction functor

$$\begin{aligned} \mathrm{ind}_{\mathcal{H}_J}^{\mathcal{H}} \colon \quad & \mathrm{Mod}(\mathcal{H}_J) \longrightarrow \mathrm{Mod}(\mathcal{H}) \\ & M_J \mapsto M_J \otimes_{\mathcal{H}_J} \mathcal{H}. \end{aligned} \tag{10}$$

Definition

Parabolic restriction is the restriction functor

$$\begin{aligned}\mathrm{Res}^{\mathcal{H}}_{\mathcal{H}_J}: \quad \mathrm{Mod}(\mathcal{H}) &\longrightarrow \mathrm{Mod}(\mathcal{H}_J) \\ M &\mapsto M \otimes_{\mathcal{H}} \mathcal{H}_J.\end{aligned} \tag{11}$$

Let (π, V) be a representation of $\mathcal{H}$. For $t \in T := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{C}^\times)$ we write

$$V_t^{\mathrm{gen}} := \left\{ v \in V \,:\, (\pi(\theta_x) - x(t))^N v = 0 \text{ for some } N \in \mathbb{N} \right\}. \tag{12}$$

When $V_t^{\mathrm{gen}} \neq 0$, we call t a *weight* of π and we denote by $\mathrm{Wt}(\pi)$ the set of weights of π.

Representations of affine Hecke algebras III

Definition

A finite dimensional representation (π, V) of $\mathcal{H}$ is *tempered* (resp. *essentially tempered*) if

$$|t(x)| \leq 1 \quad \text{for all } t \in \mathrm{Wt}(\pi),\ x \in X^+ \ (\text{resp. } x \in X^+ \cap W_{\mathrm{aff}}), \tag{13}$$

where $X^+ := \{x \in X : \langle x, \alpha^\vee \rangle \geq 0\ \forall \alpha \in \Delta\}$.

The complex torus T admits a polar decomposition

$$T = T_{\mathrm{un}} \times \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{R}_{>0}), \quad \text{where } T_{\mathrm{un}} := \mathrm{Hom}_{\mathbb{Z}}(X, \mathbb{S}^1). \tag{14}$$

Here the unitary part T_{un} is the maximal compact subgroup of T.

Representations of affine Hecke algebras IV

Definition

A *Langlands datum* for $\mathcal{H}$ is a pair (J, δ) formed by a subset J of Δ and an irreducible essentially tempered representation δ of $\mathcal{H}_J$ in "positive position".

Theorem

Let (J, δ) be a Langlands datum for $\mathcal{H}$. Then

(1) The representation $\mathrm{ind}_{\mathcal{H}_J}^{\mathcal{H}}(\delta)$ has an irreducible quotient, say $L(J, \delta)$.

(2) For every irreducible representation π of $\mathcal{H}$, there exists a Langlands datum (J, δ) such that $L(J, \delta) \simeq \pi$.

(3) If (J', δ') is another Langlands datum and $L(J', \delta') \simeq L(J, \delta)$, then $J' = J$ and the representations δ and δ' are equivalent.

Thank you so much for your attention!