Abstract Commensurators of Profinite Groups

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More Specific Questions: Given a profinite group G can it be embedded in a simple t.d.l.c. group L as an open subgroup?
If yes, is such L unique?

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A **virtual automorphism** of G is an (continuous) isomorphism between two subgroups of finite index (open subgroups).

We say that two virtual automorphisms φ and ψ are **equivalent** if they agree on a subgroup of finite index (an open subgroup). We then write $[\varphi]$ for the equivalence class of φ.

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Notice that the following is commuting:

$$\begin{array}{ccccc} H & \leq & N_G(H) & \leq & \mathrm{Comm}_G(H) \\ \downarrow & & \downarrow & & \downarrow \\ H & \rightarrow & \mathrm{Aut}(H) & \rightarrow & \mathrm{Comm}(H). \end{array}$$

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Also: If $U \leq_f G$ ($U \leq_o G$), then $\mathrm{Comm}(U) \cong \mathrm{Comm}(G)$.

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For $U \leq_o G$ let $\kappa_U : \mathrm{Aut}(U) \to \mathrm{Comm}(U) \cong \mathrm{Comm}(G)$. Then Aut-topology is the strongest topology in which κ_U is continuous for all $U \leq_o G$. We write $\mathrm{Comm}(G)_A$ for the group with the Aut-topology.

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In this case $\mathrm{Comm}(G)_A$ is a topological group. However, whether it is Hausdorff or locally compact are more refined questions which are related to the algebraic structure of G.

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Let the **virtual centre** of G be

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If $VZ(G) \leq_c G$, then $\mathrm{Comm}(G)_S$ is a locally compact group. However, it is not always σ-compact.

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6. **Neukrich and Uchida:** $G = G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \Rightarrow$ $\mathrm{Comm}(G) \cong G$.

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Corollary: Let G be a p-adic analytic pro-p group with $\mathrm{VZ}(G) = 1$, then G can be embedded as an open subgroup in at most one topologically simple t.d.l.c. group.

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Theorem: Let G be a profinite group. If G contains a non-trivial normal sticky subgroup N such that $\mathrm{Cent}_G(N) \neq 1$, then G cannot be embedded as an open subgroup of a compactly generated topologically simple t.d.l.c. group.

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Corollary: Let G be a profinite group with $\mathrm{VZ}(G) = 1$. If R_G, the fitting subgroup of G, is non-trivial and nilpotent, then G cannot be embedded as an open subgroup of a compactly generated topologically simple t.d.l.c. group.

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Example: $SL_d(\mathbb{F}_p[[X, Y]]$ cannot be embedded as an open subgroup of a compactly generated topologically simple t.d.l.c.

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Proposition: Let Γ be a residually finite discrete group. Suppose $\Gamma \leq \Delta$, where Δ is a finitely generated simple group and $\mathrm{Comm}_\Delta(\Gamma) = \Delta$. Assume $\widehat{\Gamma}$ is just infinite and $\mathrm{VZ}(\widehat{\Gamma}) = 1$. Then $\widehat{\Gamma} \leq_o \widehat{\mathrm{Comm}}(\Gamma) = \left\langle \widehat{\Gamma}, \mathrm{Comm}(\Gamma) \right\rangle$ which is a compactly generated topologically simple group.

Corollary: Let Γ be the Grigorchuk group. Then $\widehat{\Gamma}$ is embedded as an open subgroup of a compactly generated topologically simple t.d.l.c. group.

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Pf. **Class Röver:** Such Δ as above exists.