Invariant random subgroups and applications

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Plan

IRS

Definition

Γ a l.c group.

- $\text{Sub}(\Gamma) = \{\text{subgroups of } \Gamma\}$.
- Compact Chaubuty topology.
- Γ discrete

$$d(\Delta, \Sigma) = e^{-V} \qquad V = \sup_R \{[\Delta \cap B(R)] = [\Sigma \cap B(R)]\}.$$

- $\text{IRS}(\Gamma) = \mathcal{M}(\text{Sub}(\Gamma))^\Gamma$ - inv. prob. measures.

Examples of IRS

Example

- Normal: δ_N for $N \lhd \Gamma$.
- Lattice μ_Γ: $\Delta = g\Gamma g^{-1}$ g Haar random.
- Standard: w^*-limit of such.
- Sofic question: Is every IRS a limit?
- Universal: Γ_x for a p.m.p action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.
- Induction

$$\Pr\left\{\mathrm{Ind}_\Gamma^G(\Delta) \in A\right\} = \int_{G/\Gamma} \left\{\Pr \gamma^{-1}\Delta\gamma \in A\right\} d\mu(\gamma).$$

Measurable simplicity

Theorem (Stuck-Zimmer)

G - Simple Lie group $\mathbf{R}_{\mathrm{rk}}(G) \geq 2$. Non-trivial ergodic IRS:

$$(i)\ \delta_{\langle e\rangle}, \qquad (ii)\ \delta_G, \qquad (iii)\ \mu_\Gamma.$$

- Follows for lattices by induction.
- Generalizes Margulis Normal subgroup theorem.
- False for $\mathrm{SL}_2(\mathbf{R})$.

Graph Theorey,

- X_n: finite graphs $(q+1)$-regular,
- $a_\ell(n) = \#\{\text{closed non-BT paths, length } \ell \text{ in } X_n\}$.
- $\gamma_\ell = \lim_{n\to\infty} a_\ell(n)$ (on subsequence).

Theorem (Abért - G - Virág)

The following are equivalent

- $\gamma_\ell = 0 \qquad \forall \ell \in \mathbf{N}$
- $\sum_{\ell=1}^{\infty} \gamma_\ell q^{-\ell/2} < \infty$

Spectral reformulation

- X_n: finite graphs $(q+1)$-regular,
- $\mu(n) =$Spectral measure of RW operator.
- $\mu = \lim_{n\to\infty} \mu(n)$ (on subsequence).

Theorem (Abért - G - Virág)

The following are equivalent

- $\mu\left([-2\sqrt{q}, 2\sqrt{q}]\right) = 1$
- $\mu = \mu_{\mathrm{KM}} \quad = \frac{q+1}{2\pi} \frac{\sqrt{4q-x^2}}{(q+1)^2-x^2} dx$

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Limit multiplicity theorems

- X_n different, compact, G-loc-sym. (G simple $\mathbf{R}_{\text{rk}} \geq 2$).
- $X = G/K$ the symmetric space.

Theorem (Abért, Bergeron, Biringer, Gelander, Nikolov, Raimbault, Samet)

$$\frac{m(\pi, X_n)}{vol(X_n)} \to d(\pi) \qquad \forall \pi \in \hat{G}, d(\pi) = \textit{formal dimension}$$

$$\frac{b_k(X_n)}{vol(X_n)} \to \beta_k^{(2)}(X) \qquad \forall k \in \mathbf{N}$$

Realization problem for Furstenberg entropy

- $(\Gamma, \mu) \curvearrowright (X, \mathcal{B}, [\eta])$ (stationary action).
- Furstenberg entropy:

$$h_\mu(X, \eta) := \int \int -\log \frac{d\eta \circ g}{d\eta} d\eta(x) d\mu(g).$$

- **Nevo-Zimmer:** G connected $\mathbf{R}_{\mathrm{rk}} G \geq 2$. Then h_μ attains only finitely many values.

Theorem (Bowen)

For $G = F_2$ the image of h is an interval $[0, h_{max}]$.

Probability preserving actions

- $\Gamma = \langle S \rangle < \mathrm{GL}_n(F)$.
- No amenable normal subgroups.
- No commuting normal subgroups.

Theorem (G)

There exits a free subgroup $F < \Gamma$ such that
For every non-free p.m.p action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$

- $F \cdot x = \Gamma \cdot x$, *a.s.*
- $F \cap \Gamma_x$ *is non-abelian free,* *a.s.*

Tits alternative for IRS

- $\Gamma = \langle S \rangle < \mathrm{GL}_n(F)$.
- No amenable normal subgroups.
- No commuting normal subgroups.

Theorem

There exits a free subgroup $F < \Gamma$ such that
For every IRS $\langle e \rangle \neq \Delta < \Gamma$

- $F\Delta = \Gamma$, *a.s.*
- $\Delta \cap \Gamma$ *is non-abelian free,* *a.s.*

Essential subgroups

Definition (Essential subgroups)

Let $\Delta < \Gamma$ be an IRS. A subgroup $M < \Gamma$ is essential if $\Pr\{M < \Delta\} \gneqq 0$

Lemma

If $\Delta < \Gamma$ is an IRS in a countable group. Then Δ is a.s. locally essential.

Proof. Let $\Sigma_1, \Sigma_2, \ldots$ be the f.g. non-essential subgroups. $\{\Delta \in \mathrm{Sub}(\Gamma) \mid \Delta \text{ is not loc. ess. }\} = \bigcup_i \{\Delta \in \mathrm{Sub}(\Gamma) \mid \Delta > \Sigma_i\}$ and the right hand side is a countable union of null sets. $\square$

Some rep^n theory

We will assume from now on

- $\Gamma < \mathrm{GL}_{n+1}(k)$ with k local field. (Need something more general when Γ is not f.g.)
- Γ strongly irreducible on $\mathbf{P}(k^n)$
- $\mathbf{H} = \overline{\Gamma}^Z$ simple, with trivial center.
- $\exists$ strongly proximal $g \in \Gamma$.

Definition

$$\mathcal{E}^{f,SI} = \text{Stronly irreducible f.g. essential subgroups} = \{\Sigma_1, \Sigma_2, \ldots\}$$

Two central propositions

Theorem

Every non-trivial IRS $\Delta < \Gamma$ contains some $\Sigma \in \mathcal{E}^{f,SI}$ a.s.

Theorem

There exits elements:

$$\{a(i,j,\gamma) \in \Gamma \mid i,j \in \mathbf{N}, \gamma \in \Gamma\}$$

that

- *Freely generate a free group,*
- $a(i,j,\gamma) \in \Sigma_i \gamma \Sigma_j$

A baby case

We will show: The IRS $\Delta < \Gamma$ a.s. contains an infinite ess. subgroup. Assume the contrary.

- In fact condition on all essential subgroups being finite.
- Δ is locally essential a.s. $\Rightarrow$ locally finite a.s.
- $\Rightarrow \Delta$ has a finite orbit $Y = \{s_1, s_2, \dots, s_l\} \subset \mathbf{P}(k^n)$ a.s.
- $\lambda \in \Gamma$ - essential element.
- $g \in \Gamma$ very proximal element
- v, H attracting point and repelling hyperplane for g.
- w.l.o.g $Y \cap H_g = \emptyset$.

A baby case II

By Poincaré recurrence:

for almost all Δ with $\lambda \in \Delta$ we have $g^n \lambda g^{-n} \in \Delta$ for infinitely many n.

- For such subsequence $\lambda g^{n_m} Y = g^{n_m} Y$. But $g^{n_m} Y \to v$.
- Since λ, g are (almost) general. All essential elements are trivial.
- Hence Δ is trivial. $\square$