Locally Compact, Totally Disconnected Contraction Groups

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Locally Compact Groups Beyond Lie Theory
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Objects of study:

G	topological group
$\alpha\colon G \to G$	*contractive automorphism*, i.e. $\alpha^n(x) \to 1$ as $n \to \infty$, $\forall x \in G$.

Then call (G, α) a **contraction group**.

For G locally compact:

Fact (Siebert 1986). $G = G_1 \times N$, *where* $G_1 \subseteq G$ *is the connected identity component and* N *a totally disconnected,* α*-stable normal subgroup* (*i.e.,* $\alpha(N) = N$)*.*

Siebert also characterized the **connected** locally compact contraction groups

(simply connected nilpotent Lie groups admitting a positive graduation on the Lie algebra).

Now: Structure of **totally disconnected**, locally compact contraction groups

Throughout the talk, G will be **locally compact** and **totally disconnected**

Overview

§§ 3–6: joint work with George A. Willis

§1 Examples of contraction groups

Example 1 (right shift).

F	finite group
$G := F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$	$F^{(-\mathbb{N})} := \bigoplus_{n \in -\mathbb{N}} F$ discrete, $F^{\mathbb{N}_0}$ compact
$\alpha\colon G \to G$	right shift, $\alpha((x_n)_{n\in\mathbb{Z}}) := (x_{n-1})_{n\in\mathbb{Z}}$.

Then α is a contractive automorphism and hence (G, α) a contraction group.

Proof: Every 1-neighbourhood U of G contains $\mathbf{1} \times F^{\{k,k+1,\ldots\}}$ for some $k \in \mathbb{Z}$. Hence, for each $x \in G$, have $\alpha^n(x) \in U$ for large n.

Example 2 (Contractive linear automorphisms)

$\mathbb{Q}_p$	Field of p-adic numbers (tdlc)
$E := (\mathbb{Q}_p)^k$	fin-dim $\mathbb{Q}_p$-vector space
$\alpha\colon E \to E$	lin automorphism s.t. $\\|\alpha\\|_{\text{op}} < 1$ for some norm on E.

Then (E, α) is a contraction group

Example 3 (Contraction groups associated with arbitrary automorphisms)

Any automorphism α of a tdlc group G gives rise to a contraction group

$$U_\alpha := \{x \in G\colon \alpha^n(x) \to 1 \text{ as } n \to \infty\}.$$

If U_α is closed, then it is locally compact and thus $(U_\alpha, \alpha|_{U_\alpha})$ is a tdlc contraction group.

§2 Classical facts (Siebert)

If (G, α) *is a tdlc contraction group, then*:

(a) *There is a compact, open subgroup* $W \subseteq G$ *which is* α*-invariant (i.e.,* $\alpha(W) \subseteq W$*).*

Thus $\cdots \supseteq \alpha^{-1}(W) \supseteq W \supseteq \alpha(W) \supseteq \alpha^2(W) \supseteq \cdots$

(b) *For* W *as before,* $G = \bigcup_{n \in \mathbb{Z}} \alpha^{-n}(W)$

(For each $x \in G$, have $\alpha^n(x) \in W$ for some n, hence $x \in \alpha^{-n}(W)$)

In particular, G *is* σ*-compact.*

(c) *The sets* $\alpha^n(W)$ *form a basis of identity neighbourhoods. Notably,* G *is metrizable.*

(d) *If* $G \neq \mathbf{1}$*, then* G *is neither compact nor discrete*

(e) *If* $G \neq \mathbf{1}$*, then the module* $\Delta(\alpha^{-1})$ *is an integer* ≥ 2.

§3 Existence of composition series

(G,α) a tdlc contraction group

Observation (G./Willis).

The length n *of each strictly ascending series*

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of α*-stable closed subgroups* G_j *of* G *is bounded by the number* n_0 *of prime factors of* $\Delta(\alpha^{-1})$ (*counted with multiplicities*).

Hence there exist **composition series** of α-stable closed subgroups

(series which cannot be refined further).

Fact. *Let G be a locally compact group and $\alpha \in \mathrm{Aut}(G)$. If $N \subseteq G$ is an α-stable closed normal subgroup and α' the automorphism of G/N induced by α, then*

$$\Delta_G(\alpha) = \Delta_N(\alpha|_N) \cdot \Delta_{G/N}(\alpha').$$

Proof (for observation)

Let α_j be the automorphism of $Q_j := G_j/G_{j-1}$ induced by α. Then

$$\Delta_G(\alpha^{-1}) = \Delta_{Q_1}(\alpha_1^{-1}) \cdot \ldots \cdot \Delta_{Q_n}(\alpha_n^{-1}),$$

where each module is an integer ≥ 2
$\rightsquigarrow$ assertion. $\square$

Defn. A contraction group (G, α) is called **simple** if $G \neq \mathbf{1}$ and G does not have α-stable closed normal subgroups except for $\mathbf{1}$ and G.

Remark A series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

of α-stable closed subgroups is a composition series if and only if all factors G_j/G_{j-1} are simple contraction groups (with respect to the automorphism induced by α).

Observation. *Any simple contraction group* (G, α) *is either abelian or topologically perfect, i.e.,* $\overline{[G, G]} = G$. *If* G *is abelian, then* G *is either torsion free or a torsion group of prime exponent.*

Proof. $N := \overline{[G, G]}$ is an α-stable closed normal subgroup of G. Hence $N = \mathbf{1}$ or $N = G$.

If G is abelian and $x^p = 1$ for some $x \neq 1$, then $\{g \in G \colon g^p = 1\}$ is a closed α-stable non-trivial subgroup and hence coincides with G. □

§4 Main results

Classification of the simple totally disconnected contraction groups

Theorem (G./Willis).

If (G, α) is a simple tdlc contraction group, then G is either a torsion group or torsion free.

Classification:

(a) *If G is a torsion group, then (G, α) is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift, for some finite simple group F.*

(b) *If G is torsion free, then (G, α) is isomorphic to $(\mathbb{Q}_p)^d$ with a $\mathbb{Q}_p$-linear contractive automorphism for which there are no invariant vector subspaces.*

Conversely, all of these are simple.

The classification has consequences for the structure of arbitrary tdlc contraction groups (G, α)

Structure Theorem (G./Willis).

The set tor(G) *of torsion elements and the set* div(G) *of divisible elements are fully invariant closed subgroups of* G *and*

$$G = \text{tor}(G) \times \text{div}(G)\,.$$

Moreover, tor(G) *has finite exponent and*

$$\text{div}(G) = G_{p_1} \times \cdots \times G_{p_n}$$

is a direct product of certain α*-stable* p*-adic Lie groups* G_p*.*

Remark. By J.S.P. Wang (1984), each G_p is nilpotent, and in fact the group of $\mathbb{Q}_p$-rational points of a unipotent linear algebraic group defined over $\mathbb{Q}_p$.

§5 Proof of the classification

Case $(G,+)$ is abelian and a torsion group of prime exponent p:

Pick $1 \neq x$ and set $F := \langle x \rangle \cong C_p$. Then

$$F^{(-\mathbb{N})} \times F^{\mathbb{N}_0} \to G, \quad (x_n)_{n\in\mathbb{Z}} \mapsto \sum_{n=N}^{\infty} \alpha^n(x_n)$$

(with N so small that $x_n = 0$ for all $n < N$) defines a homomorphism of groups that intertwines the right shift and α (and can be shown to be a topological isomorphism)

To tackle the torsion–free case, need another basic tool for tdlc contraction groups (G, α):

Tool 1 (G./Willis).

If $W \subseteq G$ is a closed subgroup with $\alpha(W) \subseteq W$, then

$$S := \bigcup_{n\in\mathbb{N}_0} \alpha^{-n}(W)$$

is an α-stable **closed** *subgroup of G.*

Hence, if G is simple, W normal and $W \neq \mathbf{1}$, then $S = G$ and W is open in G.

Classification if $(G, +)$ *is abelian and torsion free*:

Let $W \subseteq G$ be a compact, open subgroup such that $\alpha(W) \subseteq W$. Then W can be chosen pro-p for some p.

[For some prime p, there exists a p-Sylow subgroup $P \neq \mathbf{1}$ of W, which is unique as W is abelian. Hence $\alpha(P) \subseteq P$ and thus, by Tool 1, P is open. Now replace W with P if necessary.]

For $0 \neq x \in W$, can define zx for $z \in \mathbb{Z}_p$. Let $W(x)$ be the image of the continuous homomorphism

$$\phi\colon \mathbb{Z}_p^{\mathbb{N}_0} \to W, \quad (z_n)_{n\in\mathbb{N}_0} \mapsto \sum_{n\in\mathbb{N}_0} \alpha^n(z_n x).$$

Then $W(x)$ is a compact, non-trivial α-invariant subgroup of G and hence open by Tool 1.

Being a torsion-free abelian pro-p-group, $W(x)$ is isomorphic to $\mathbb{Z}_p^J$ for some set J.

(This can be shown using Pontryagin duality).

Since $pW(x) = W(px)$ is open, $(p\mathbb{Z}_p)^J$ must be open in $\mathbb{Z}_p^J$, whence J is finite and $W(x) \cong \mathbb{Z}_p^J$ a p-adic Lie group.

Now a linearization argument shows that

$$(G, \alpha) \cong (L(G), L(\alpha)).$$

Classification of non-abelian simple contraction groups uses a further tool:

Tool 2 (G./Willis).

If $W \subseteq G$ is a closed non-trivial subgroup with $\alpha(W) \subseteq W$, then its core

$$N := \bigcap_{g \in G} gWg^{-1}$$

is a **non-trivial** *closed, α-invariant normal subgroup of G.*

Tool 1 implies:

If G is simple, then N is open in G.

Case (G,α) *simple, non-abelian*

Have $gh \neq hg$ for suitable $g,h \in G$. By Tool 2, G has an α-invariant, compact, open, **normal** subgroup W. We may assume that $g \in W$ and choose $m \in \mathbb{N}$ so large that $hgh^{-1}g^{-1} \notin \alpha^m(W)$. Then $\pi(h) \neq 1$ for the permutation representation

$$\pi\colon G \to \mathrm{Sym}(W/\alpha^m(W)),$$

$\pi(x)(y\alpha^m(W)) := xyx^{-1}\alpha^m(W)$, and thus $\ker(\pi)$ is a proper closed normal subgroup of G of finite index. It is therefore contained in a maximal normal subgroup M. Then $F := G/M$ is a finite simple group. Let $q\colon G \to G/M = F$ be the quotient map. One now verifies that

$$G \to F^{(-\mathbb{N})} \times F^{(\mathbb{N}_0)}, \quad x \mapsto (q(\alpha^{-n}(x)))_{n\in\mathbb{Z}}$$

is an isomorphism of topological groups which intertwines α and the right shift.

§6 Outline of proof for the Structure Theorem

Know (G, α) admits a composition series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G.$$

I. Easy case: If all factors G_j/G_{j-1} are torsion, then G is a torsion group (of finite exponent, and hence without divisible elements)

II. Not too hard: If all factors G_j/G_{j-1} are torsion-free and hence p-adic, then G is a direct product of p-adic Lie groups for some primes p, and is divisible.

III. The composition series can be chosen such that the torsion factors appear first.

Thus tor$(G) = G_k$ for some k and G/G_k is a product of p-adic Lie groups.

IV. Construct a complement for G_k.

§7 Contractible Lie groups over local fields

$\mathbb{K}$ a local field (tdlc, non-discrete)

An analytic Lie group G over $\mathbb{K}$ is called **contractible** if it admits a contractive analytic automorphism $\alpha\colon G \to G$.

Theorem (G.) *Let G be a contractible Lie group over a local field $\mathbb{K}$. If* char$(\mathbb{K}) > 0$*, then G is a torsion group.*

Idea of proof. By the Structure Theorem, $G = \text{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}$ with p-adic Lie groups G_p. Show that each $G_p = \mathbf{1}$. □

As in the p-dic case (treated by Wang), always have:

Theorem (G.) *Every contractible Lie group over a local field is nilpotent.*

Analytic proof, which uses ideas from the theory of time-discrete, analytic **dynamical systems**:

Idea of proof: Let $\alpha\colon G \to G$ be contractive. For $a \in \,]0,1]$, consider the **a-contraction group** U_a which comprises all $x \in G$ such that

$$\|\phi(\alpha^n(x))\| \;=\; o(a^n)$$

for large n, where ϕ is a chart for G around 1 with $\phi(1) = 0$ and $\|.\|$ is a norm on the modelling space for G.

U_a is a submanifold of G (for suitable a), a so-called **a-stable manifold** for the dynamical system (G, α) around the fixed point 1
(and a closed Lie subgroup; cf. G., Expo Math).

Since $[U_a, U_b] \subseteq U_{ab}$, one can find $a_1 < a_2 < \cdots < a_n$ such that

$$\mathbf{1} \;=\; U_{a_1} \;\subseteq\; U_{a_2} \;\subseteq\; \cdots \;\subseteq\; U_{a_n} \;=\; G$$

is an ascending central series for G. Hence G is nilpotent.

Remark The invariant manifolds just described can also be used to calculate the scale* $s_G(\alpha)$ of an analytic automorphism α of an analytic Lie group G over a local field, provided that U_α is closed (see G. [7, 8]).

The p-adic case is easier, because U_α is always closed here (see Wang) and the exponential function can be used to linearize the dynamical system. See [3] for the calculation of the scale in the p-adic case (and also for linear automorphisms of $\mathbb{K}^n$); for semisimple groups over local fields, see Baumgartner and Willis [1].

Remark Further general information on contraction groups can also be found in Baumgartner and Willis [1] (extended to non-metrizable groups by Jaworski [10], cf. [4]), including discussions of closedness of U_α. Compare also Müller-Römer [11]. Contraction groups of p-adic groups were also studied by Dani and Shah [2].

*In the sense of Willis [14, 15]

§8 Summary

Several results are now known about the structure of tdlc contraction groups (G, α):

- G has a composition series;
- The composition factors are unique up to topological isomorphism and order;
- The possible composition factors can be classified;
- Structure Theorem:

  $G = \text{tor}(G) \times \text{div}(G) = \text{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}$

- Application: Structure of contractible Lie groups over local fields (notably in positive characteristic)

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