Independence conditions for group actions on trees

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Property (P)

Let G be a group acting on some tree T.

Definition

For a path L in T define $\mathrm{pr}_L : VT \to VT$ such that $\mathrm{pr}_L(v)$ is the vertex on L that is closest to v.
For a vertex x in L denote with $G^x_{(L)}$ the permutation group induced by $G_{(L)}$ on $\mathrm{pr}_L^{-1}(x)$.
The action is said to have property P if the homomorphism

$$G_{(L)} \to \prod_{x \in L} G^x_{(L)}$$

is an isomorphism for all paths L (finite or infinite) in T.

Tits's Theorem

Theorem

(Tits, 1970) Let T be a tree and G a subgroup of $\text{Aut}(T)$. Assume that G does not stabilize a proper subtree, and that G does not fix an end of T. Assume furthermore that the action of G on T has property P. Then G^+, the subgroup of G generated by stabilizers of edges, is simple.

Property H

Let G be a group acting on a tree T.
We say the action has property H if the stabilizer of every half-tree in T is non-trivial.

Simplicity result.

Theorem

(M.&V., 2012) Let T be a tree and G a closed subgroup of $\mathrm{Aut}(T)$*. Assume that G does not stabilize a proper subtree, and that G does not fix an end of T. Assume furthermore that the action of G on T has property H. Then G^{++}, the subgroup of G generated by pointwise stabilizers of half-trees, is topologically simple.*

If N is a non-trivial subgroup normalized by G^{++} then $G^{++} \leq N$.

Quasi-center

Definition

The quasi-centre $QZ(G)$ of G is the subgroup consisting of all elements that have an open centralizer.

Theorem

(cf. Caprace and De Medts, 2011) Let T be a tree and G a closed subgroup of $\mathrm{Aut}(T)$. Assume that G does not stabilize a proper subtree. Assume furthermore that the action of G on T has property H. Then the quasi-center of G is trivial.

Now on to something completely different
Primitive graphs

Definition

A group G is said to act primitively on a set Y if no non-trivial proper equivalence relation on Y is preserved by the action.

Definition

A graph X is said to be primitive if the automorphism group acts primitively on the vertex set VX.

Graphs with connectivity 1

The connectivity of a non-complete graph X is the least number of vertices that one needs to remove so that what is left of the graph is not connected.
A *block* of X is a maximal 2-connected subgraph (i.e. connected and does not have connectivity 1).
"A 1-connected graph is made up of its blocks that are joined in a tree like fashion."

Primitive graphs with connectivity 1

Theorem

(Jung & Watkins, 1977) Let X be an infinite graph with connectivity 1. Then X is primitive if and only if all the blocks are primitive and pairwise isomorphic and have at least three vertices and each vertex belongs to the same number of blocks.

Digression: The oddness of the number 2

Theorem

(Jung & Watkins, 1989) There exists no infinite, locally finite, primitive graph whose connectivity equals 2.

Connectivities equal to $1, 3, 4, \ldots$ are all possible.

The importance of graphs with connectivity 1

Theorem

(M., 1994) Suppose X is a locally finite primitive graph with infinitely many ends and $G = \mathrm{Aut}(X)$. Then there exist vertices u, v in X such that the graph $Y = (VX, G\{u, v\})$ has connectivity 1 and each block of Y has at most one end.

Acessibility of primitive graphs

A graph X is said to be *accessible* if there is a number k such that any two ends can be separated by removing at most k vertices.
The graph Y is clearly accessible and X is quasi-isometric to X and hence X is accessible.

Theorem

(M., 1994) A primitive locally finite graph is always accessible.

Group theoretic version

Using the construction of Abels-Cayley graphs one gets:

Theorem

Let G be a compactly generated totally disconnected locally compact group. If G has a maximal subgroup that is compact and open then G is accessible.

Infinite stabilizers

Theorem

(Simon Smith, 2006) If G acts primitively on a locally finite graph with more than one end then the vertex stabilizers are infinite.

Simplicity for automorphism groups of primitive graphs

Theorem

(M.&V., 2012) Let G be the automorphism group of a locally finite primitive graph X with more than one end. Then G has a topologically simple subgroup of finite index.