Finitely presented groups with infinitely many asymptotic cones

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Definition

Let

- μ *a non principal ultrafilter;*
- $\hat{X} = (X_n, d_n)_{n\in\mathbb{N}}$ *a sequence of metric spaces;*
- $\hat{e} = (e_n)_{n\in\mathbb{N}}$ *a sequence of points with* $e_n \in X_n$*.*

Set

$$\mathcal{F} = \{(x_n)_{n\in\mathbb{N}}, x_n \in X_n \mid \exists c,\ s.t.\ d_n(e_n, x_n) \leq c, \forall n \in \mathbb{N}\},$$

$$(x_n) \sim (y_n) \text{ iff } \lim_{\mu} d_n(x_n, y_n) = 0, \quad d_\mu((x_n),(y_n)) = \lim_{\mu} d_n(x_n, y_n).$$

Then $(\mathcal{F}/\sim, d_\mu)$ *is a metric space called the ultralimit of the sequence* $\hat{X}$ *relative to* $\mu, \hat{e}$*.*

Definition

Let G be a group with a finite generating set S and $(C_G(S), d)$ the Cayley graph of G with respect to S. Let

- *μ a non principal ultrafilter;*
- *$\hat{\lambda} = (\lambda_n)_{n\in\mathbb{N}}$ a sequence of non negative real numbers with $\lim_\mu \lambda_n = +\infty$;*
- *$\hat{e} = (e_n)_{n\in\mathbb{N}}$ a sequence of points of $C_G(S)$.*

The ultralimit of $\hat{X} = (C_G(S), d/\lambda_n)_{n\in\mathbb{N}}$ relative to μ, $\hat{\lambda}$ and $\hat{e}$ is called the asymptotic cone of G relative to μ, $\hat{\lambda}$ and $\hat{e}$ and it is denoted by $Con^\mu(G, \hat{\lambda})$.

Remarks

- $\mathrm{Con}^{\mu}(G,\hat{\lambda})$ is independent on the choice of the generating set S: if S_1 and S_2 are two finite generating sets of G then the corresponding asymptotic cones are bi-Lipschitz isomorphic.
- $\mathrm{Con}^{\mu}(G,\hat{\lambda})$ is also independent on the choice of $\hat{e}$: if $\hat{e}_1$ and $\hat{e}_2$ are two sequences of G then the corresponding asymptotic cones are homeomorphic.

Hence we fix S and we take $\hat{e} = (e_n)_{n\in\mathbb{N}}$ to be the sequence with $e_n = 1$ for all $n \in \mathbb{N}$.

Roughly speaking:

Geometric properties of asymptotic cones of G

$$\Updownarrow$$

Algebraic properties of G

- G is hyperbolic iff every asymptotic cone of G is a real tree.
- G is virtually nilpotent iff every asymptotic cone of G is proper.
- If some asymptotic cone of G has cut points then an ultrapower of G contains a non abelian free group.

Question (Gromov, 1993): Is there a finitely generated (finitely presented) group with 2 non-homeomorphic asymptotic cones?

Theorem (Thomas-Velicovic, 2000)

There exists a finitely generated group with 2 non-homeomorphic asymptotic cones.

Theorem (Drutu-Sapir, 2005)

There exists a finitely generated group with $2^{\aleph_0}$ pairwise non-homeomorphic asymptotic cones.

Theorem (Kramer-Shelah-Tent-Thomas, 2005)

There exists a finitely presented group G such that:

- *If CH holds then G has a unique asymptotic cone, up to homeomorphisms,*
- *If CH fails then G has $2^{2^{\aleph_0}}$ asymptotic cones, up to homeomorphisms.*

Theorem (Ol'Sahnskii-Sapir, 2007)

There exists a finitely presented group with 2 non-homeomorphic asymptotic cones.

Theorem (Osin-O., 2011)

There exists a finitely presented group H such that

- *H has infinitely many asymptotic cones up to homeomorphisms.*
- *There exists $\hat{a} = (a_n)$ and $\hat{b} = (b_n)$ such that for any μ, $Con^{\mu}(H, \hat{a})$ has cut points while $Con^{\mu}(H, \hat{b})$ does not.*

Definition

Let X be a path-connected topological space. The connectivity degree of X, denoted $C(X)$, is defined by

$$C(X) = \min\{|D| \mid D \text{ is finite and } X \setminus D \text{ is not path-connected}\}$$

if at least one of such D exists, otherwise $C(X) = \infty$.

Remark. X has cut points iff $C(X) = 1$.

Theorem (Ol'Shanskii-Osin-Sapir, 2008)

Let G be a finitely generated group and N a central subgroup of G. Suppose that $Con^{\mu}(N, \hat{\lambda}))$ has exactly m points. Then

$$c(Con^{\mu}(G, \hat{\lambda})) = m \times c(Con^{\mu}(G/N, \hat{\lambda}).$$

Suppose that we have found a finitely presented group H and a sequence of sequences $\hat{\lambda}_i$ such that:

- $\mathrm{Con}^\mu(Z(H), \hat{\lambda}_i))$ has exactly m_i points;
- $H/Z(H)$ is constricted; that is any asymptotic cone of $H/Z(H)$ has cut points, in other words for any μ, for any $\hat{\lambda}$, $c(\mathrm{Con}^\mu(H/Z(H), \hat{\lambda})) = 1$.

Then, by the previous theorem, $c(\mathrm{Con}^\mu(H, \hat{\lambda}_i)) = m_i$ and thus $\mathrm{Con}^\mu(H, \hat{\lambda}_i)$ is not homeomorphic to $\mathrm{Con}^\mu(H, \hat{\lambda}_j)$ whenever $m_i \neq m_j$.

Theorem (O., 2003)

Every recursively presented group G embeds in a finitely presented group Γ such that $Z(G) = Z(\Gamma)$.

Theorem

Let G be a recursively presented group. Then G embeds in a finitely presented group Γ such that:

- $Z(G) = Z(\Gamma)$.
- $\forall g \in G$, $\sqrt{|g|_G} \leq |g|_\Gamma \leq |g|_G$.
- $\Gamma/Z(\Gamma)$ *is constricted.*

Let $\alpha : \mathbb{N}^2 \to \mathbb{N}$ be a recursive bijection and $\alpha_1(m) = j$ whenever $m = \alpha(j, n)$.
Let $(p_j)_{j\in\mathbb{N}}$ be a recursively enumerable sequence of primes and set

$$k_n = p_{\alpha_1(n)}.$$

Let $(R_n)_{n\in\mathbb{N}}$ be a recursively enumerable sequence of $F = \langle a, b \mid \rangle$ satisfying $C'(1/25)$ (and some technical conditions).
Let

$$G = \langle a, b \mid R_n^{k_n} = [R_n, a] = [R_n, b] = 1,\ n \in \mathbb{N}\rangle.$$

Then

- G is recursively presented.
- $Z(G) = \langle R_n,\ n \in \mathbb{N}\rangle$.

Embeds G in a finitely presented group Γ as in the previous theorem. In particular we have

$$Z(G) = Z(\Gamma)$$

$$c(\mathrm{Con}^{\mu}(\Gamma / Z(\Gamma), \hat{\lambda})) = 1, \forall \mu, \forall \hat{\lambda}$$

$$\forall g \in G, \sqrt{|g|_G} \leq |g|_\Gamma \leq |g|_G$$

Lemma

For $j \in \mathbb{N}$ let $\hat{r}_j = (r_n^j)_{n\in\mathbb{N}}$

$$r_n^j = |R_{\alpha(j,n)}|_\Gamma.$$

Then $\forall \mu$, $Con^\mu(Z(\Gamma), \hat{r}_j)$ consist exactly of p_j points.

Proof (Idea).

- The order of $R_{\alpha(j,n)}$ is p_j.
- If $(g_n)^\mu \neq (1)^\mu$, $g_n = R_1^{\varepsilon_1} \cdots R_{i_n}^{\epsilon_{i_n}}$, then $(g_n)^\mu = (R_{i_n}^{\varepsilon_{i_n}})^\mu$.
- $i_n = \alpha(j,n)$ for μ-almost n and $\exists s, 1 \leq s \leq p_j$ s.t. $(g_n)^\mu = (R^s_{\alpha(j,n)})^\mu$.

$\square$

Conclusion:

- $c(\mathrm{Con}^{\mu}(\Gamma, \hat{r}^j)) = p_j$,
- if $p_i \neq p_j$ then $\mathrm{Con}^{\mu}(\Gamma, \hat{r}^i)$ is not homeomorphic to $\mathrm{Con}^{\mu}(\Gamma, \hat{r}^j)$.

Lemma

Let $\hat{d} = (d_n)_{n\in\mathbb{N}}$

$$d_n = nk_n||R_n||.$$

Then for any μ*,* $Con^{\mu}(Z(\Gamma), \hat{d})$ *consists of exactly 1 point.*

Conclusion: $c(\mathrm{Con}^{\mu}(\Gamma, \hat{d})) = 1$.

THANKS