Tame topological dynamics.

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Aims and motivation I

- The aim is to develop a theory of the action of a *definable* group G (i.e. a group first order definable in some first order structure M) on a compact Hausdorff space.
- We would like this this theory to be *analogous* to the classical theory (G is a topological group). And a direct *generalization* of the case when G is a discrete group.
- As a *definable group*, namely first order structure $(\mathbb{F}, \cdot)$, a noncommutative free group $\mathbb{F}$ will be "uniquely extremely amenable".
- Likewise any compact Lie group G, as a group definable in $(\mathbb{R}, +, \times)$ will be "uniquely amenable".

Aims and motivation II

- We are heavily influenced by recent and current work of Newelski which tried to use the existing notions of topological dynamics (minimal flow,..) to extend "stable group theory" to tame unstable environments, but we want, in addition, to develop an intrinsic theory of actions of definable groups on compact spaces.
- A group G definable in a structure M comes equipped with its (G-invariant) Boolean algebra $Def_G(M)$ of subsets of G definable (with parameters) in M.
- If you don't know model theory, you can view our theory as about a group G equipped with a distinguished G-invariant Boolean subalgebra $\mathcal{B}$ of $\mathcal{P}(G)$ (power set of G). Our theory reduces to the classical topological dynamics of the discrete group G when $\mathcal{B}$ is the full power set $\mathcal{P}(G)$.

The classical theory I

- Topological dynamics concerns the understanding of a topological group G through its (left) actions on compact (Hausdorff) spaces X, and the case where G is discrete is important. Call (X, G) a G-flow.
- Among the basic results, objects, invariants, are:
- - existence of a unique universal G-ambit $(U(G), G, e)$ (where a G-ambit is a G-flow (X, G) with a distinguished point $x \in X$ such that $G \cdot x$ is dense), as well as a canonical (Ellis) semigroup structure on $U(G)$ (continuous in the first coordinate).
- - existence and uniqueness (up to G-isomorphism) of a universal minimal G-flow, $(M(G), G)$: any minimal subflow, equivalently minimal left ideal, of $(U(G), G)$ will do the job.

The classical theory II

- - The group of automorphisms $Aut(M(G))$ of the minimal G-flow, has a compact T_1 topology (and there is a corresponding Galois theory).
- Amenability of the (topological) group G means the existence of a G-invariant regular probability measure on $U(G)$, and extreme amenability of G means the existence of a fixed point in $U(G)$, equivalently the triviality of some (any) minimal G-flow.
- No locally compact (in particular discrete) group is extremely amenable.
- When G is discrete, $U(G)$ is simply the space βG of ultrafilters on $\mathcal{P}(G)$, under the natural left action of G, and the distinguished dense orbit is G itself, identified with the set of principal ultrafilters.

The classical theory III

- The "nonstandard analysis" account of the semigroup structure on βG is relevant: we have $*$-versions of everything, so $G^* \supset G$, $X^* \supset X$ for $X \subseteq G \times .. \times G$, etc.
- Given $p, q \in \beta G$, let $b \in G^*$ "realize" q, and let $a \in G^*$ "realize" p such that for each $Y \subseteq G \times G$, $a \in Y_b^*$ iff $\{g \in G : g \in Y_b^*\} \in p$. Then $p \cdot q$ is the ultrafilter determined by $a \cdot b$.
- Let M be a minimal (closed) subflow of $U(G)$, and $u \in M$ an idempotent (with respect to $\cdot$). Then $u \cdot M$ is a group under $\cdot$, which, acting on the right on M via $\cdot$, coincides with $Aut(M)$.
- The T_1, compact topology on $Aut(M)$ discussed in the literature (so-called τ-topology) is a bit obscure.

Model theory and the tame version I

- I don't want to give an introduction to model theory. I mentioned a way of looking at what is going on in my second slide.
- Let now M be a first order structure and $(G, \cdot)$ a group definable in M. For example M could be simply $(G, \cdot)$ itself.
- Let $B_G(M)$ be the Boolean algebra of definable, in M with parameters, subsets of G, and $S_G(M)$ the space of ultrafilters on $B_G(M)$, what we call the "space of complete types over M, concentrating on G" (a profinite space).
- We want a theory of "definable" actions of the definable group G on compact spaces, such that $(S_G(M), G, e)$ is the universal definable G-ambit (as in the discrete case).

Model theory and the tame version II

- For X a definable set in M and C a compact Hausdorff space we say that a function $f : X \to C$ is *definable* if for any disjoint closed subsets C_1, C_2 of C, $f^{-1}(C_1)$ and $f^{-1}(C_2)$ are separated by a definable set.
- Already with this (uncontroversial) definition, given a definable group G in M there is a universal definable group compactification of G, namely a definable homomorphism $f : G \to C$ with C a compact group, which is universal with respect to such maps.
- Given a definable group G and compact space X we define a group action of G on X (by homeomorphisms) to be *definable* if the corresponding map from G to X^X is definable, equivalently, for each $x \in X$ the map $G \to X$ taking g to $g \cdot x$ is definable.
- This is a rather strong and possibly controversial definition but suits our purposes.

Model theory and the tame version III

- Back to our group G definable in M. To get a smooth theory analogous to the topological case we need some additional closure properties of $B_G(M)$, specifically that for any definable subset X of G and $p \in S_G(M)$, $\{g \in G : g \cdot X \in p\}$ is definable.
- In fact for convenience we will assume more (and much more than needed), namely "definability of all types over M": For any definable set Y in M, every $p \in S_Y(M)$ is *definable*, meaning that for any definable W and definable $Z \subseteq Y \times W$, $\{b \in W : Z_b \in p\}$ is definable.
- The *stable* complete theories T are characterized by all types over all models of T being definable.
- All types over $(\mathbb{R}, + \cdot)$ and $(\mathbb{Q}_p, +, \cdot)$ are definable, although the first order theories of these two structures are unstable.

Model theory and the tame version IV

- If T is a so-called NIP theory (to be discussed later), and M a model of T, then there is a minimal "expansion" of M (obtained by declaring new subsets to be definable), M^{Sh}, such that all types over M^{Sh} are definable, and moreover $Th(M^{Sh})$ is also NIP (so tame..)
- For an NIP theory, model M of T, and group G definable in M, various properties of G are preserved under passing to M^{Sh}. So really no harm to work inside M^{Sh} instead.
- So really our theory is supposed to generalize stable group theory to the NIP context.
- In any case given the above definitions and assumptions we have the soft:

Theorem 0.1

(i) There is a unique definable G-ambit and it is precisely the space $S_G(M)$ with distinguished element $1 \in G$.
(ii) $S_G(M)$ has an Ellis semigroup structure $\cdot$, continuous in the first coordinate, defined as above.
(iii) There is a unique (up to isomorphism of definable G-flows) minimal definable G-flow $M(G)$ which can be taken as some (any) minimal subflow of $S_G(M)$, and $Aut(M(G)) = u \cdot M(G)$ for any idempotent $u \in M$, acting on $M(G)$ on the right via $\cdot$.
(iv) Also Pestov's characterization of extreme amenability goes through in the definable category: namely G is definably extremely amenable iff whenever $X \subseteq G$ is definable and "left generic" (i.e. left syndetic, i.e. finitely many left translates cover G), then $XX^{-1} = G$

Stable groups I

- We mean M is a model of a stable theory, and G a group definable in M.
- This is a well-developed theory; stable group theory or equivariant stability theory, although the (trivial) translation into topological dynamics language was only seen more recently.
- There is a unique minimal subflow $M(G)$ of $S_G(M)$, and it is the space of "generic types", on which G acts equicontinuously,
- Moreover $Aut(M(G))$ with its τ topology is a compact Hausdorff (in fact profinite) group, isomorphic to the definable group compactification $G^*/(G^*)^0$ of G, and its action on $M(G)$ is regular.
- In fact $(M(G), \cdot)$ is already a compact group.

Stable groups II

- G is definably (uniquely) amenable, and is definably extremely amenable iff G is "connected" (no proper definable subgroup of finite index).
- Examples of stable groups are groups definable in stable fields: algebraically closed fields (algebraic groups), differentially closed fields (differential algebraic groups),... as well as any abelian group $(G, +)$ considered as a structure in just the group language.
- An important class of new and surprizing examples are torsion-free hyperbolic groups $(G, \cdot)$ in the group language (Sela).
- In particular a noncommutative free group is stable, and connected, hence uniquely extremely amenable.

Generalizations I

- There are a number of contexts outside stable theories where aspects of stable group theory apply: "simple" theories, NIP theories, pseudofinite theories,... In fact "pseudofinite theories" is the environment for the results relevant to approximate subgroups.
- As far as our formalism developed above is concerned, we feel that NIP theories are the right context.
- A first order theory T is NIP if every definable family of definable sets has Vapnik-Chervonenkis dimension: given a family $X_b : b \in Y$ of definable sets, there is a bound on the size of finite sets A (in models of T) such that every subset of A is of the form $X_b \cap A$ for some $b \in Y$.
- Stable theories are NIP and examples of unstable NIP theories include RCF, the first order theory of the p-adic field, and the theory of algebraically closed valued fields.

Generalizations II

- Newelski more or less conjectured that in the NIP context and with our definitions and assumptions on M, $Aut(M(G))$ is a compact Hausdorff group isomorphic to the definable group compactification $G^*/(G^*)^{00}$ of G.
- I first discuss an analysis of the case where $M = (\mathbb{R}, +, \cdot)$ and $G = SL_2(\mathbb{R})$, yielding in particular a counterexample.
- So we consider $SL_2(\mathbb{R})$ as a definable group rather than topological group (or discrete group). Its definable group compactification is trivial.
- Let V be G/H where H is the connected component of the group of upper triangular matrices in G.
- Let X be $S_V(M) \setminus V$. Then X with the natural action of G is the universal definable minimal G-flow, and $Aut(X)$ is naturally isomorphic to $Z(G)$ (so nontrivial).

Generalizations III

- $SL_2(\mathbb{R})$ is not definably amenable.
- So we refine the Newelski conjecture to the case where G is definably amenable.
- At this point the model theory becomes more delicate, so I more or less just consider examples.
- There are two extreme ways (and many other ways) that G can be definably amenable: (i) existence of finite satisfiable generics, (ii) existence of definable generics, and one would like to reduce definable amenability to these two forms.
- In both cases, the conjecture holds.

Generalizations IV

- In (i) there is a unique minimal subflow of $S_G(M)$, the generic types, as in the stable case, and also a unique G-invariant Borel probability measure on $S_G(M)$. But the action of G on the universal minimal flow is in general not even distal.
- The typical example is where G is a compact Lie group definable in $(\mathbb{R}, +, \times)$, but considered as a definable group. The definable group compactification of G is itself, but now as a compact group, so in a sense recovering the topology from the Boolean algebra of definable sets.
- In (ii) there could be several minimal subflows of $S_G(M)$ (and several G-invariant probability measure on $S_G(M)$), but the action of G on each minimal subflow of $S_G(M)$ is equicontinuous.
- One of the simplest examples of (ii) is where $G = M = (\mathbb{Z}, +, <)$, so-called Presburger arithmetic.

Generalizations V

- All types over M are definable so this example fits into our assumptions.
- There are two minimal subflows of $S_G(M)$, at plus infinity and minus infinity.
- The automorphism group of each of them is $\hat{\mathbb{Z}}$ which coincides with the definable group compactification of G.
- Identical analysis for the multiplicative group of the p-adics (as a group definable in $(\mathbb{Q}_p, +, \cdot)$).

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