Simple Groups of Automorphisms of Trees

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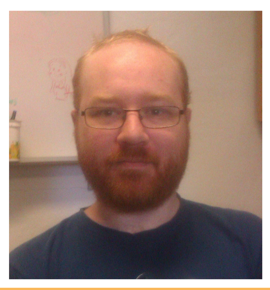

Abstract

Groups of automorphisms of trees $G < \mathrm{Aut}(\mathcal{T})$ are investigated and results proved about their simplicity by Jacques Tits [5]; these results used an Independence Property (P). We describe methods of constructing groups acting on trees with slightly weaker properties, and use this framework to prove similar simplicity results.

The k-closure of G

Let $\mathcal{T}$ be a tree and G a group acting on $\mathcal{T}$. Let $B(v,k)$ denote the closed ball (with respect to the standard metric on $\mathcal{T}$) of radius k centred at the vertex $v \in V(\mathcal{T})$. Then for each $k \in \mathbb{N}$, define the *k-closure of G* to be

$$G^{(k)} = \{\alpha \in \mathrm{Aut}(\mathcal{T}) \mid \forall v \in V(\mathcal{T}),\ \exists\, g \in G \text{ such that } g|_{B(v,k)} = \alpha|_{B(v,k)}\}.$$

Properties

- $G^{(k+1)} \leq G^{(k)} \leq \mathrm{Aut}(\mathcal{T})$ for all k.
- $G^{(k)}$ is closed and $\bigcap G^{(k)} = \overline{G}$, the closure of G.
- The orbits $G^{(k)}.v = G.v$ are equal for all vertices v.
- Two subgroups G, H satisfy $G^{(k)} = H^{(k)}$ if for all $v \in V(\mathcal{T})$ we have $\mathrm{Stab}_G v|_{B(v,k)} = \mathrm{Stab}_H v|_{B(v,k)}$ and $G.v = H.v = (G \cap H).v$.

Theorem

Let $G \leq \mathrm{Aut}(\mathcal{T})$, fix $k \in \mathbb{N}$ and suppose that G does not leave invariant any proper subtree of $\mathcal{T}$. Then $G^{(k)}$ is non-discrete if and only if there is $(v,w) \in E(\mathcal{T})$ and $g \in G$ such that

$$g|_{B(v,k)\cap B(w,k)} = 1 \ \text{ and } \ g|_{B(w,k)} \neq 1.$$

Examples

- Each $G^{(1)}$ contains any group acting with the same orbits and the same local actions as G. If G is vertex-transitive and locally transitive (in which case $\mathcal{T}$ is regular), the *universal group* construction [2] fully describes the structure of $G^{(1)}$.
- All 7 vertex-transitive locally transitive discrete groups acting on $\mathcal{T}_3$ [3] satisfy $G^{(3)} = G$, and in 3 of these cases $G^{(2)} \neq G$.
- When all k-closures of G are distinct, as is the case for $G = PSL(2,\mathbb{Q}_p)$ acting on $\mathcal{T}_{p+1}$, they form an infinite descending series of subgroups whose intersection is G.

Example: Automorphism groups of graphs

Let Γ be any graph with universal covering tree $\mathcal{T}$. Then $\pi_1(\Gamma)$ acts naturally on $\mathcal{T}$ and there exists $G < \mathrm{Aut}(\mathcal{T})$ forming the exact sequence

$$\pi_1(\Gamma) \hookrightarrow G \twoheadrightarrow \mathrm{Aut}(\Gamma).$$

For Γ a finite graph, G is a discrete subgroup of $G^{(k)}$, and $G^{(k)} = G$ for all $k \geq \mathrm{diam}(\Gamma)$. The graph $C(p,r,1)$ [4] on the right, has vertex set $\{(i,k) : i \in \mathbb{Z}_r, 1 \leq j \leq p\}$ and adjacency $(i,k)\ (j,l)$ iff $j = i \pm 1$. Here the k-closure of $G_{p,r}$ is non-discrete for all $k < \frac{r}{2}$. For distinct r, s the k-closures of $G_{p,r}$ and $G_{p,s}$ are the same for all $k < \frac{r}{2}, k < \frac{s}{2}$.

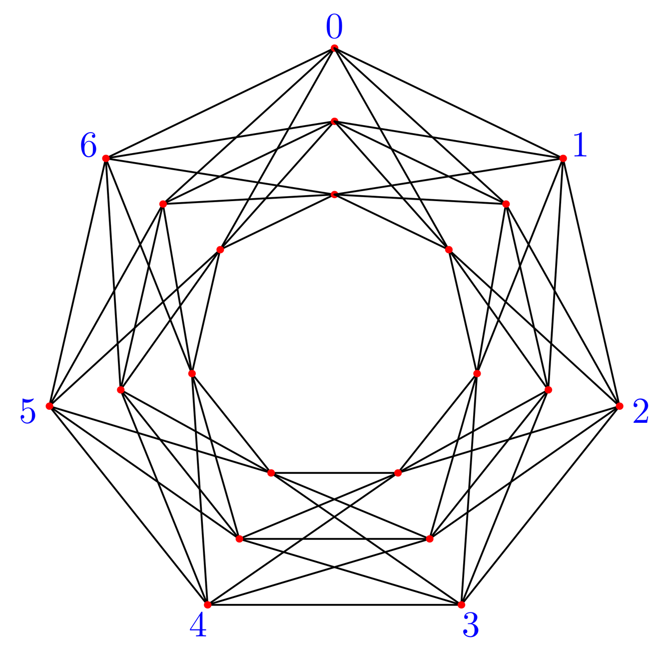

Figure: The graph $C(3,7,1)$

The Independence Properties: generalising the work of Tits

Definition (Property (P^k))

Suppose $G < \mathrm{Aut}(\mathcal{T})$ and fix $k \in \mathbb{N}$. For any edge $\{v,w\}$, let $\mathcal{T}_{(v,w)}$ denote the semitree of $\mathcal{T}$ containing w but not $\{v,w\}$, and let $\mathcal{B}_k$ be the vertex set $B(v,k) \cap B(w,k)$. Let $F := \mathrm{Fix}_G(\mathcal{B}_k)$ be the subgroup of G fixing $\mathcal{B}_k$. Then G satisfies *Property* (P^k) if $F \cong \mathrm{Fix}_F(\mathcal{T}_{(v,w)})\mathrm{Fix}_F(\mathcal{T}_{(w,v)})$ for all edges.

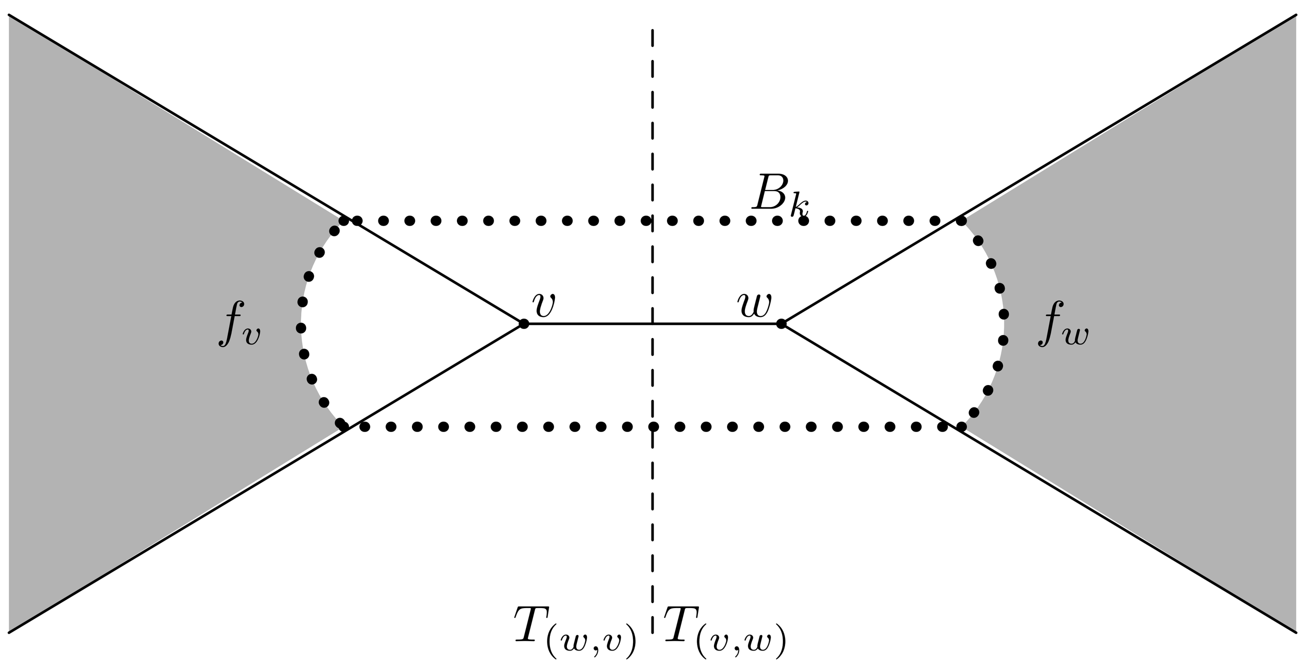

Figure: Property (P^k); f_v fixes $\mathcal{T}_{(v,w)}$ and f_w fixes $\mathcal{T}_{(w,v)}$. Both act independently of each other, and any $f \in \mathrm{Fix}_G(\mathcal{B}_k)$ splits into some product $f_v f_w$.

Properties of Properties (P^k)

- (P^1) is equivalent to Independence Property (P) [5].
- Property (P^k) implies Property (P^j) for all $j > k$.
- $G^{(k)}$ has Property (P^k), and $H^{(k)} = \overline{H}$ iff H has Property (P^k).

Simplicity Theorem

Let $G < \mathrm{Aut}(\mathcal{T})$ be closed, fix $k \in \mathbb{N}$ and let G^{+_k} denote the group generated by automorphisms in $\mathrm{Fix}_G(\mathcal{B}_k)$ for any edge of $\mathcal{T}$. Suppose that G satisfies Property (P^k) and does not stabilise a proper non-empty subtree or an end of $\mathcal{T}$. Then every nontrivial subgroup of G normalised by G^{+_k} contains G^{+_k}; in particular G^{+_k} is simple or trivial.

References

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