Gelfand pairs for locally compact groups acting on Euclidean buildings

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Abstract

Let $\mathbf{G}$ be a locally compact group acting properly by type-preserving automorphisms on a Euclidean building $\mathbf{\Delta}$, and $\mathbf{G_{x_0}}$ be the stabilizer of a special vertex $\mathbf{x_0} \in \mathbf{\Delta}$. It is known that $(\mathbf{G}, \mathbf{G_{x_0}})$ is a Gelfand pair as soon as $\mathbf{G}$ acts strongly transitively on $\mathbf{\Delta}$; this is in particular the case when $\mathbf{G}$ is a semi-simple group over a local field. We show a converse to this statement, namely: if $(\mathbf{G}, \mathbf{G_{x_0}})$ is a Gelfand pair and $\mathbf{G}$ acts cocompactly on $\mathbf{\Delta}$, then the action is strongly transitive. The proof uses the existence of strongly regular elements in $\mathbf{G}$ and their peculiar dynamics on the spherical building at infinity.

Introductory main definitions

- We say $\mathbf{G}$ acts **strongly transitively** on $\mathbf{\Delta}$ (resp. $\partial\mathbf{\Delta}$) if for every two pairs $(\mathbf{A}, \mathbf{C_1})$ and $(\mathbf{B}, \mathbf{C_2})$, with $\mathbf{A}, \mathbf{B}$ two apartments in $\mathbf{\Delta}$ (resp. $\partial\mathbf{\Delta}$) and $\mathbf{C_1}, \mathbf{C_2}$ two chambers with $\mathbf{C_1} \in \mathrm{Ch}(\mathbf{A})$ and $\mathbf{C_2} \in \mathrm{Ch}(\mathbf{B})$, there exists $\mathbf{g} \in \mathbf{G}$ such that $\mathbf{g}(\mathbf{B}) = \mathbf{A}$ and $\mathbf{g}(\mathbf{C_2}) = \mathbf{C_1}$;
- For a l.c. group $\mathbf{G}$ and $\mathbf{K} < \mathbf{G}$ compact let $\mathbf{C_c^K(G)}$ be the set of all compact supported, $\mathbf{K}$-bi-invariant continuous, complex-valued functions on $\mathbf{G}$ endowed with the convolution product.
The pair $(\mathbf{G}, \mathbf{K})$ is said to be **Gelfand** if the convolution algebra $\mathbf{C_c^K(G)}$ is commutative.

Main Theorem

Let $\mathbf{\Delta}$ be a locally finite thick Euclidean building and $\mathbf{x_0} \in \mathbf{\Delta}$ be a special vertex. Let $\mathbf{G}$ be a closed, non-compact, type-preserving subgroup of $\mathrm{Aut}(\mathbf{\Delta})$ without fixed point in $\partial\mathbf{\Delta}$. Then the following are equivalent:

i) $\mathbf{G}$ acts strongly transitively on $\mathbf{\Delta}$;
ii) $\mathbf{G}$ acts strongly transitively on $\partial\mathbf{\Delta}$;
iii) the $\mathbf{G}$-action on $\mathbf{\Delta}$ is cocompact and there exists an apartment $\mathbf{A}$ of $\mathbf{\Delta}$ such that for every $\mathbf{c} \in \mathrm{Ch}(\partial\mathbf{A})$ the subgroup $\mathbf{G_c} \lneqq \mathbf{G}$ has the property that $\mathbf{G}/\mathbf{G_c}$ is compact;
iv) the $\mathbf{G}$-action on $\mathbf{\Delta}$ is cocompact and there exists a compact open subgroup $\mathbf{K} < \mathbf{G}$ having a finite number of chamber-orbits on $\partial\mathbf{\Delta}$;
v) $\mathbf{G}$-action on $\mathbf{\Delta}$ is cocompact and $(\mathbf{G}, \mathbf{G_{x_0}})$ is a Gelfand pair.

Key ingredients in the proof

- existence of strongly regular hyperbolic elements and their dynamics on the spherical building $\partial\mathbf{\Delta}$;
- fundamental equivalence: '$\mathbf{G}$ acts strongly transitively on $\mathbf{\Delta}$' $\iff$ '$\mathbf{G}$ acts strongly transitively on $\partial\mathbf{\Delta}$';
- playing with two non-$\mathbf{G_{x_0}}$-conjugate strongly regular hyperbolic elements in $\mathbf{G}$ to prove: '$\mathbf{G}$ not strongly transitive on $\partial\mathbf{\Delta}$' $\implies$ '$(\mathbf{G}, \mathbf{G_{x_0}})$ not a Gelfand pair'.

Strongly regular elements and their dynamics

- $\mathbf{a} \in \mathrm{Aut}(\mathbf{\Delta})$ is strongly regular hyperbolic element if it admits a unique translation apartment $\mathbf{A_a}$ in $\mathbf{\Delta}$ and whose translation axes are not parallel with any of the walls of the apartment $\mathbf{A_a}$;
- for every $\xi \in \partial\mathbf{\Delta}$ the limit $\lim\limits_{\mathbf{n}\to\infty} \mathbf{a^n}(\xi)$ exists in $\partial\mathbf{A_a}$ (w.r.t. the cone topology) and the fixed-point-set of $\mathbf{a}$ in $\partial\mathbf{\Delta}$ is $\partial\mathbf{A_a}$.

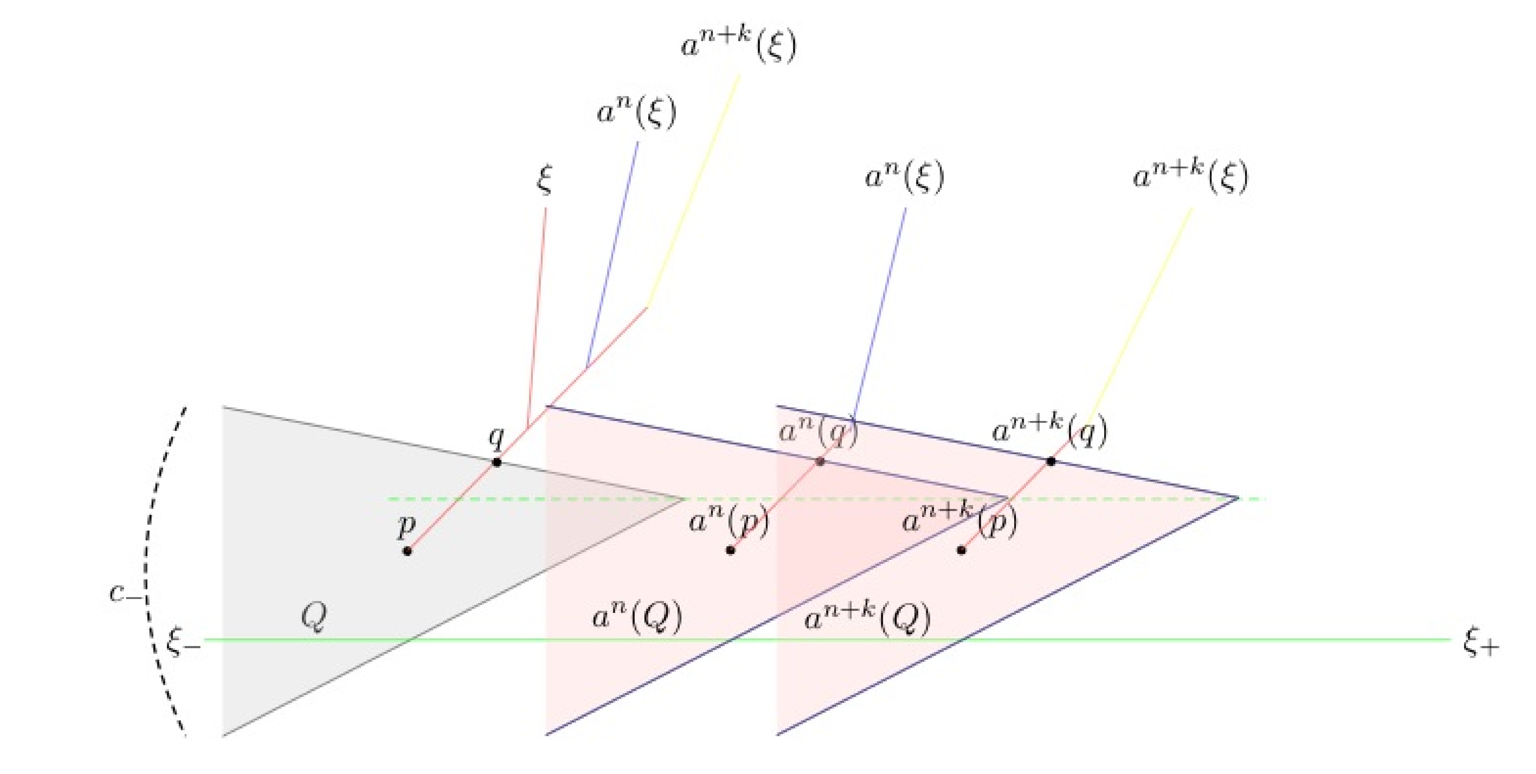

Figure 1: the green line (ξ_-, ξ_+) is a translation axis of the strongly regular element a; c_- is the chamber in ∂A_a containing ξ_-; Q is a sector in A_a corresponding to c_-; the red line between p and ξ is the geodesic ray $[p, \xi)$.

Fundamental equivalence

- '$\mathbf{G}$ acts strongly transitively on $\mathbf{\Delta}$' $\implies$ '$\mathbf{G}$ acts strongly transitively on $\partial\mathbf{\Delta}$' is well known;
- '$\mathbf{G}$ acts strongly transitively on $\partial\mathbf{\Delta}$' $\implies$ '$\mathbf{G}$ acts strongly transitively on $\mathbf{\Delta}$'
Sketch:
  - every apartment in $\mathbf{\Delta}$ has cocompact stabilizer in $\mathbf{G}$;
  - 'Geometric' Levi decomposition: For every pair $\mathbf{c}, \mathbf{c'}$ of opposite chambers in $\mathrm{Ch}(\partial\mathbf{\Delta})$, we have $\mathbf{G_c} = \mathbf{G_{c,c'}}.\mathbf{G_c^0}$. In particular $\mathbf{G_c^0}$ is transitive on the set of chambers opposite $\mathbf{c}$;
  - criterion of strong transitivity on $\mathbf{\Delta}$.

Remark: The fundamental equivalence is used to prove ii) $\iff$ iii) $\iff$ iv) in the main Theorem.

Sketch of proof v) $\implies$ ii)

Suppose '$\mathbf{G}$ is not strongly transitive on $\partial\mathbf{\Delta}$'.
Steps to follow:

- the number of $\mathbf{G_{x_0}}$-chamber-orbits in $\partial\mathbf{\Delta}$ is infinite;
- construct two $\mathbf{a} \neq \mathbf{b}$ non-$\mathbf{G_{x_0}}$-conjugate strongly regular hyperbolic elements in $\mathbf{G}$;
- apply the dynamics of strongly regular elements to obtain 'good' powers of $\mathbf{a}$ and $\mathbf{b}$ and 'good' properties;
- define $\mathbf{f} := \mathbf{1}_{\mathbf{G_{x_0} a^m G_{x_0}}}, \mathbf{g} := \mathbf{1}_{\mathbf{G_{x_0} b^n G_{x_0}}} \in \mathbf{C_c^{G_{x_0}}(G)}$, for $\mathbf{n}, \mathbf{m} > \mathbf{0}$ 'good' powers founded above;
- calculate

$$\mathbf{f} * \mathbf{g}(\mathbf{a^m b^n}) = \int_{\mathbf{G}} \mathbf{f}(\mathbf{a^m b^n h})\mathbf{g}(\mathbf{h^{-1}})\mathbf{d}\mu(\mathbf{h})$$
$$\geq \int_{\mathbf{b^{-n}G_{x_0}}} \mathbf{f}(\mathbf{a^m b^n h})\mathbf{g}(\mathbf{h^{-1}})\mathbf{d}\mu(\mathbf{h}) = \mu(\mathbf{K}) > \mathbf{0};$$

- show that $\mathbf{g} * \mathbf{f}(\mathbf{a^m b^n}) = \int_{\mathbf{G}} \mathbf{g}(\mathbf{a^m b^n h})\mathbf{f}(\mathbf{h^{-1}})\mathbf{d}\mu(\mathbf{h}) = \mathbf{0}$.

Contradiction with $\mathbf{C_c^{G_{x_0}}(G)}$ is commutative.

Further applications: Flat closing conjecture on general buildings

Strongly regular hyperbolic elements acting on Euclidean buildings are used to give a positive answer to **Gromov's flat closing conjecture** on general buildings:

- Let $\mathbf{\Delta}$ be a locally finite thick building of type $(\mathbf{W}, \mathbf{S})$, with $\mathbf{S}$ finite. Let $\mathbf{G} \leq \mathrm{Aut}(\mathbf{\Delta})$ a type-preserving subgroup with cocompact action. Does the existence of a $\mathbf{d}$-flat $\mathbf{F} \subset \mathbf{\Delta}$ imply that $\mathbf{G}$ contains a copy of $\mathbb{Z}^{\mathbf{d}}$?

Conclusions

For totally disconnected locally compact groups the only known examples of Gelfand pairs are coming from groups acting by automorphisms on Euclidean buildings. More precisely in [Léc10] and [APVM11] they prove that for $\mathbf{G}$ a closed, strongly transitive and type-preserving subgroup of $\mathrm{Aut}(\mathbf{\Delta})$, with $\mathbf{\Delta}$ a locally finite thick building, there is no Gelfand pair if $\mathbf{\Delta}$ non-Euclidean. If $\mathbf{\Delta}$ is Euclidean there is a Gelfand pair $(\mathbf{G}, \mathbf{K})$, with $\mathbf{K}$ a parabolic subgroup, only in the well known case when $\mathbf{K} = \mathbf{G_x}$, with $\mathbf{x}$ a special vertex in $\mathbf{\Delta}$.

References

[APVM11] P. Abramenko, J. Parkinson, and H. Van Maldeghem, *A classification of commutative parabolic Hecke algebras*, 2011. arXiv **1110.6214**.

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