Anosov diffeomorphisms on infra-nilmanifolds

Jonas Deré

Introduction

Infra-nilmanifolds play a very important role in dynamical systems, especially when studying expanding maps or Anosov diffeomorphisms. Because of the algebraic nature of these manifolds, questions about self-maps can be translated into questions about endomorphisms of their fundamental groups. In this way, it was shown in [1] by M. Gromov that every expanding map on a compact manifold is topologically conjugate to an affine infra-nilendomorphism. Up to now it is unknown if a similar statement also holds for Anosov diffeomorphisms, although some partial results point in that direction. This question of classifying all Anosov diffeomorphisms on compact manifolds up to homeomorphism was already raised by S. Smale in 1969, see [5]. This motivates my research, which is about classifying infra-nilmanifolds admitting an Anosov diffeomorphism.

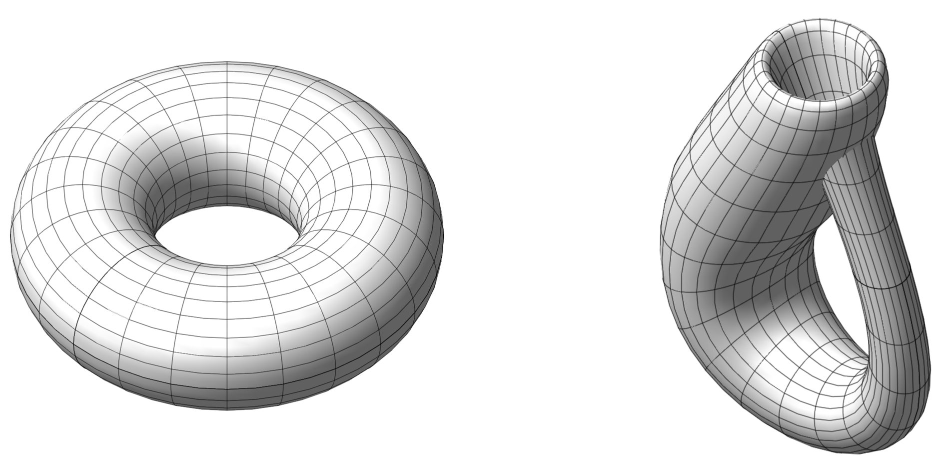
What is an infra-nilmanifold?

Let G be a connected and simply connected nilpotent Lie group. The affine group Aff$(G) = G \rtimes$ Aut(G) acts on G in the following natural way:

$$\forall \alpha = (g, \delta) \in \text{Aff}(G), \forall h \in G : {}^{\alpha}h = g\delta(h).$$

Let $F \subseteq$ Aut(G) be a finite group of automorphisms of G and $\Gamma \subseteq G \rtimes F$ be a discrete and torsion-free subgroup such that the quotient $\Gamma\backslash G$ is compact. Such a group Γ is called an almost Bieberbach group (AB group) and the quotient space $\Gamma\backslash G$ is a compact manifold which we call an infra-nilmanifold. Let $p : \Gamma \to F$ be the natural projection on the second component, then we call $p(\Gamma)$ the holonomy group of $\Gamma\backslash G$.

Example. *If G is abelian, i.e. if G is the additive group $\mathbb{R}^n$ for some n, then infra-nilmanifolds are just compact flat manifolds, including the Klein bottle and all tori:*



The class of infra-nilmanifolds coincides with the class of almost flat manifolds.

What is an Anosov diffeomorphism?

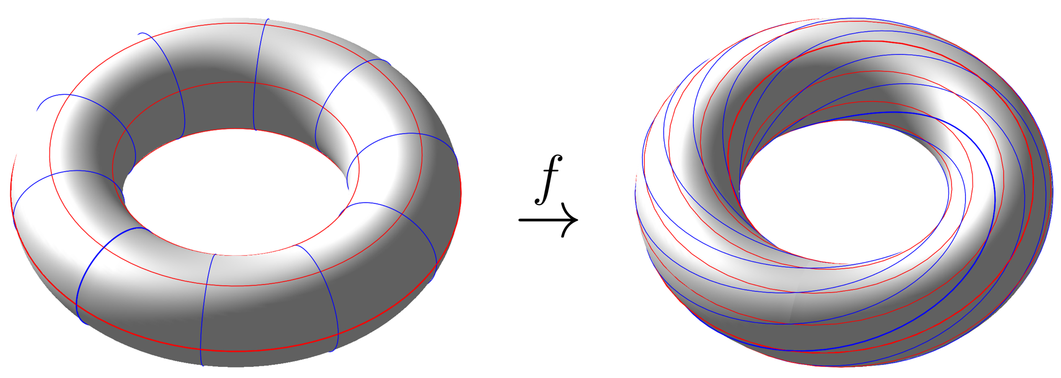
Let M be a compact manifold and $f : M \to M$ a C^1-diffeomorphism. We call f an Anosov diffeomorphism if there exists a continuous splitting $TM = E^s \oplus E^u$ and a Riemannian metric on M such that:

(i) The subbundles E^s and E^u are preserved under the map $Tf : TM \to TM$.

(ii) There exists a constant $0 < \lambda < 1$ such that

$$\forall v \in E^u, \forall n \in \mathbb{N} : \|Tf^n(v)\| \geq \lambda^{-n}\|v\|,$$
$$\forall v \in E^s, \forall n \in \mathbb{N} : \|Tf^n(v)\| \leq \lambda^{n}\|v\|.$$

Example. *Consider the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, which induces a diffeomorphism f on the 2-torus $T^2 = \mathbb{Z}^2\backslash\mathbb{R}^2$. This map f is the well-known Arnold's cat map:*



More generally, every hyperbolic matrix $A \in$ GL$(n, \mathbb{Z})$ induces an Anosov diffeomorphism on the n-torus.

Why study Anosov diffeomorphisms on infra-nilmanifolds?

Definition. *Let $\Gamma \subseteq$ Aff(G) be an AB group and $\alpha \in$ Aff(G) such that $\alpha\Gamma\alpha^{-1} = \Gamma$. The affine transformation α induces a diffeomorphism on the infra-nilmanifold $\Gamma\backslash G$, which is called an affine infra-nilautomorphism.*

Note that an affine infra-nilautomorphism is an Anosov diffeomorphism if and only if the linear part of α is hyperbolic. The examples of Anosov diffeomorphisms above were all constructed in this way.

The following conjecture motivates the study of Anosov diffeomorphism on infra-nilmanifolds:

Conjecture. *Every Anosov diffeomorphism on a compact manifold M is topologically conjugate to an affine infra-nilautomorphism.*

The conjecture is known to be true in the case of infra-nilmanifolds [2] and in the case of codimension one Anosov diffeomorphisms [3].

Rational holonomy representation

Every AB group Γ with holonomy group F fits in a short exact sequence

$$1 \to N \to \Gamma \to F \to 1,$$

where $N = \Gamma \cap G$ is the subgroup of pure translations. By embedding the group N in its rational Mal'cev completion $N_{\mathbb{Q}}$, we find the following commutative diagram of groups

$$\begin{array}{ccccccccc} 1 & \to & N & \to & \Gamma & \to & F & \to & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \to & N_{\mathbb{Q}} & \to & \Gamma_{\mathbb{Q}} & \to & F & \to & 1, \end{array}$$

where the last exact sequence splits. By fixing a splitting morphism $s : F \to \Gamma_{\mathbb{Q}}$, we define the rational holonomy representation $\varphi : F \to$ Aut$(N_{\mathbb{Q}})$ by

$$\varphi(f)(n) = s(f)ns(f)^{-1}.$$

It turns out that the rational holonomy representation contains all information about the existence of Anosov diffeomorphisms.

Main result: classification for infra-nilmanifolds modeled on a free nilpotent Lie group

Let $\varphi : F \to$ Aut$(N_{\mathbb{Q}})$ be the rational holonomy representation of an AB group Γ. By looking at the abelianized rational holonomy representation

$$\bar{\varphi} : F \to \text{Aut}(N_{\mathbb{Q}}/[N_{\mathbb{Q}}, N_{\mathbb{Q}}]) \cong \text{GL}(n, \mathbb{Q}),$$

we are able to use all the methods of representation theory for finite groups. The following theorem extends an important result [4] of Porteous from the abelian case to the class of free nilpotent Lie groups:

Theorem. *Let $\Gamma\backslash G$ be an infra-nilmanifold with G a free nilpotent Lie group of rank c and $\bar{\varphi} : F \to$ GL$(n, \mathbb{Q})$ the abelianized rational holonomy representation. Then the following are equivalent:*

$\Gamma\backslash G$ admits an Anosov diffeomorphism.

$\Updownarrow$

Every $\mathbb{Q}$-irreducible component of $\bar{\varphi}$ that occurs with multiplicity m, splits in more than $\frac{c}{m}$ components when seen as a representation over $\mathbb{R}$.

References

[1] Gromov, M. *Groups of polynomial growth and expanding maps.* Institut des Hautes Études Scientifiques, 1981, 53, pp. 53–73.

[2] Manning, A. *There are no new Anosov diffeomorphisms on tori.* Amer. J. Math., 1974, 96 3, pp. 422–429.

[3] Newhouse, S. E. *On codimension one Anosov diffeomorphisms.* Amer. J. Math., 1970, 92 761–770.

[4] Porteous, H. L. *Anosov diffeomorphisms of flat manifolds.* Topology, 1972, 11, pp. 307–315.

[5] Smale, S. *Differentiable dynamical systems.* Bull. Amer. Math. Soc., 1967, 73, pp. 747–817.

Email: jonas.dere@kuleuven-kulak.be

The author is supported by a Ph. D. fellowship of the Research Foundation – Flanders (FWO) and by the Research Fund KU Leuven.