Constructing Non-Trivial Words

in Non-Periodic Branch Groups

This is an example of a branch group G that has exponential growth but does not contain any non-abelian free subgroups. We demonstrate how to construct a non-trivial word $w_{a,b}(x,y)$ for any $a,b \in G$ such that $w_{a,b}(a,b)=1$. The group G is not just infinite. We prove that every normal subgroup of G is finitely generated as an abstract group and every proper quotient soluble. Further, G has infinite virtual first Betti number but is not large.

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Branch groups

Rooted Trees:

- sequence $\{l_i\}$ of coprime integers
- vertices on level n have l_n vertices below

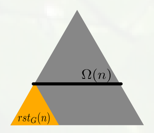
Automorphisms

automorphism on T: map that preserves edge incidence and the root r.

Subgroups

$st_G(n)$ fixes all vertices on and above level n

$rst_G(n)$ fixes all vertices outside the subtree with root v, a vertex on level n



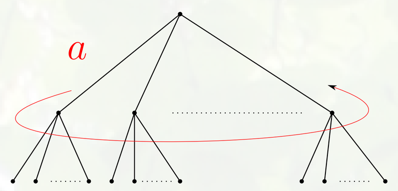
Branch Groups

1. act transitively on each level of the tree
2. $rst_G(n)$ has finite index in G for all $n \geq 0$

The Group And Subgroups

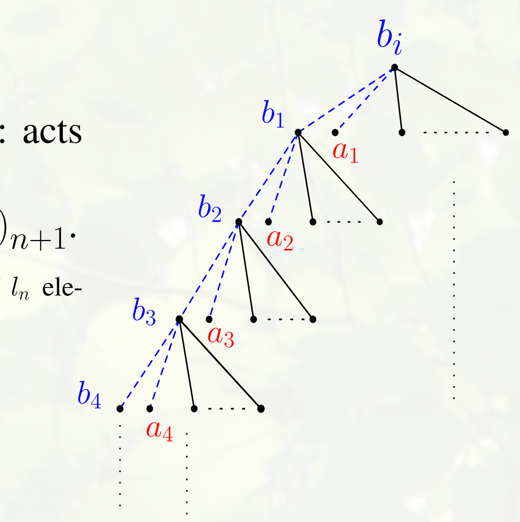
The Generators

- a rooted automorphism a: permutes the l_0 vertices on level 1 cyclically



- a spinal automorphism b: acts on each level $n+1$ as
$b_n = (b_{n+1}, a_{n+1}, 1, \ldots, 1)_{n+1}$.

where a_n is a cyclic permutation of l_n elements



The Structure

Define $b(i) = b^{a^{i-1}}$.

- $G' = \langle b(2)^{-1}b(1), b(3)^{-1}b(2), \ldots, b(1)^{-1}b(l_0)\rangle$.

Define $G_n = \langle a_n, b_n \rangle$

- $G_n' \times \cdots \times G_n' = G^{(n+1)} \leq G$
- $G^{(n+1)} \leq rst_G(n)$

Some Properties

- G is a branch group.
- every proper quotient of G is soluble.
- every normal subgroup of G is finitely generated
- G has infinite virtual first Betti number
- G is not just infinite
- G is not large

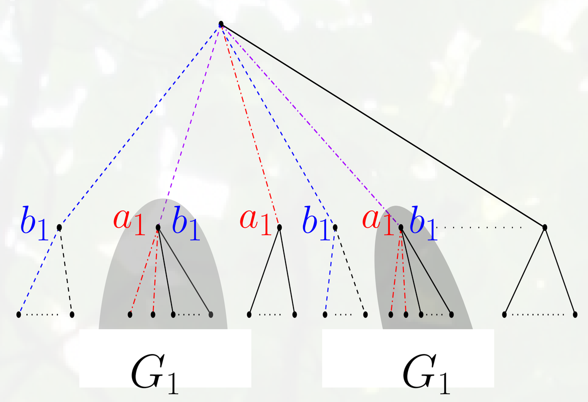
The Growth

Theorem. *If the sequence $\{l_i\}$ satisfies that each l_i is prime and that*

$$\log(l_i - 1) \geq 5 \cdot \left(\frac{47}{5}\right)^i \prod_{j=0}^{i-1} l_j$$

for all i then the 2-generator group G has ***exponential growth*** *rate.*

The proof of this Theorem uses the following observation:

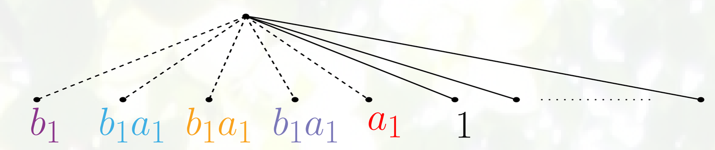


Finding Words

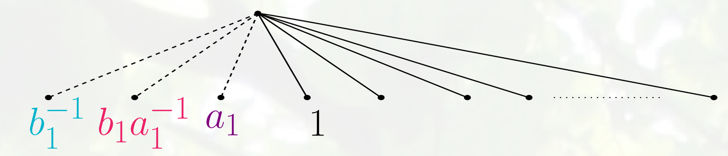
An Example

Let g_1 and g_2 be in G. We construct a word $w_{g_1,g_2}(x,y) \in F_2$ such that $w_{g_1,g_2}(g_1,g_2) = 1$ in G.

$$g_1 = (ab)^4a^{-4} = (b_1, b_1a_1, b_1a_1, b_1a_1, a_1, 1, \ldots, 1)_1$$

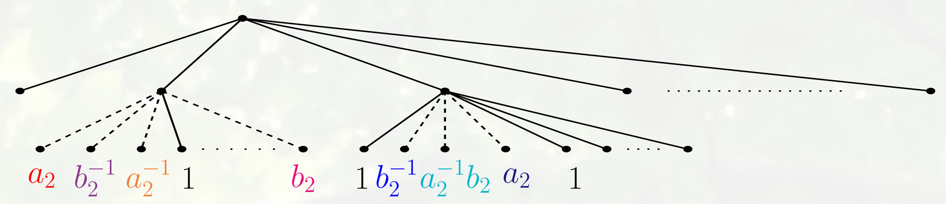


$$g_2 = [b, b^a] = \left(b_1^{-1}, b_1a_1^{-1}, a_1, 1, \ldots, 1\right)_1$$



Step 1: $c_1 = [g_1, g_2]$

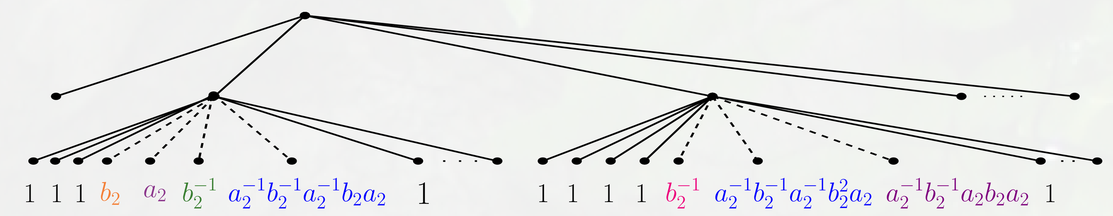
$$c_1 = \left(1, \left[b_1a_1, b_1a_1^{-1}\right], [b_1a_1, a_1], 1, \ldots, 1\right)_1 =$$
$$\left(1, \left(a_2, b_2^{-1}, a_2^{-1}, 1, \ldots, b_2\right), \left(1, b_2^{-1}, a_2^{-1}b_2, a_2, 1, \ldots, 1\right), 1, \ldots, 1\right)_2$$



We see that above every non-trivial entry on level 2 we have the element a_1b_1 in g_1 which is of the form $a_1^q b$, with $q \neq 0$.

We assume l_2 is big enough and move the spines away from their original position:

$$c_1^{g_1^4} = \left(1, \left(1,1,1,b_2,a_2,b_2^{-1}, a_2^{-1}b_2^{-1}a_2^{-1}b_2a_2, 1, \ldots, 1\right),\right.$$
$$\left.\left(1,1,1,1,b_2^{-1}, a_2^{-1}b_2^{-1}a_2^{-1}b_2^2a_2, a_2^{-1}b_2^{-1}a_2b_2a_2, 1, \ldots, 1\right), 1, \ldots, 1\right)_2.$$



This tree and the tree for c_1 on the left only decorate distinct vertices. Hence

$$\left[c_1, c_1^{g_1^4}\right] = 1.$$

This gives the word

$$w_{g_1,g_2}(x,y) = \left[[x,y],[x,y]^{x^4}\right] =$$
$$y^{-1}x^{-1}yx^{-3}y^{-1}x^{-1}yx^4y^{-1}xyx^{-5}y^{-1}xyx^4.$$

The Theorem

Let $g_1, g_2 \in G$. Recursively define commutators

$$c_1 = [g_1, g_2] \quad \text{and} \quad c_i = \left[c_{i-1}, c_{i-1}^{c_{i-2}}\right], \text{ for } i \geq 2 \text{ with } c_0 = g_1.$$

Lemma 1. *If $g_1, g_2 \in G$, then the number of spines $\xi(c_i)$ in the commutator c_i defined as above is bounded by $\xi(c_i) \leq 5^i(\xi(g_1) + \xi(g_2))$ for all $i \geq 0$.*

The strategy is to observe that the number of spines of the commutators c_i grows more slowly than the number of vertices on each level. We note the position of the spines of c_i and aim to move them by conjugation such that none of the conjugated spines is at an old position. This new element will then commute with c_i.

Lemma 2. *For every $i \geq 1$ we have $c_i \in rst_G(i)$.*

Commutators have a recursive form such that we can always find some power of a 'above' every non-trivial entry.

Theorem. *Assume that the defining sequence $\{l_i\}$ satisfies $l_i \geq (25l_{i-1})^{3\prod_{j=0}^{i-1} l_j}$. Then G has no free subgroup of rank 2.*

The proof of Theorem at least allows slight improvements on the bound for l_i. However, it seems likely that the construction of a non-trivial word will require the sequence l_i to grow at least as fast as $l_i \geq C \cdot l_{i-1}^{D \cdot m_i}$ where C and D are non-zero constants.

Private communication with Nekrashevych suggests that the result of his can be used to prove the absence of free subgroups for all defining sequences $\{l_i\}$.