Subgroup structure of the Gupta–Sidki group

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Motivation

Groups acting on rooted trees are the subject of intense research. They provide answers to many hard problems in group theory: Burnside problem, Day problem (amenable but not elementary amenable groups), Milnor problem (groups of intermediate growth).
Some of the most well-known examples of these types of group are the Grigorchuk group and the Gupta–Sidki p-groups.
We study two aspects of their subgroup structure.

Commensurability Two groups are (abstractly) **commensurable** if they have isomorphic finite index subgroups.

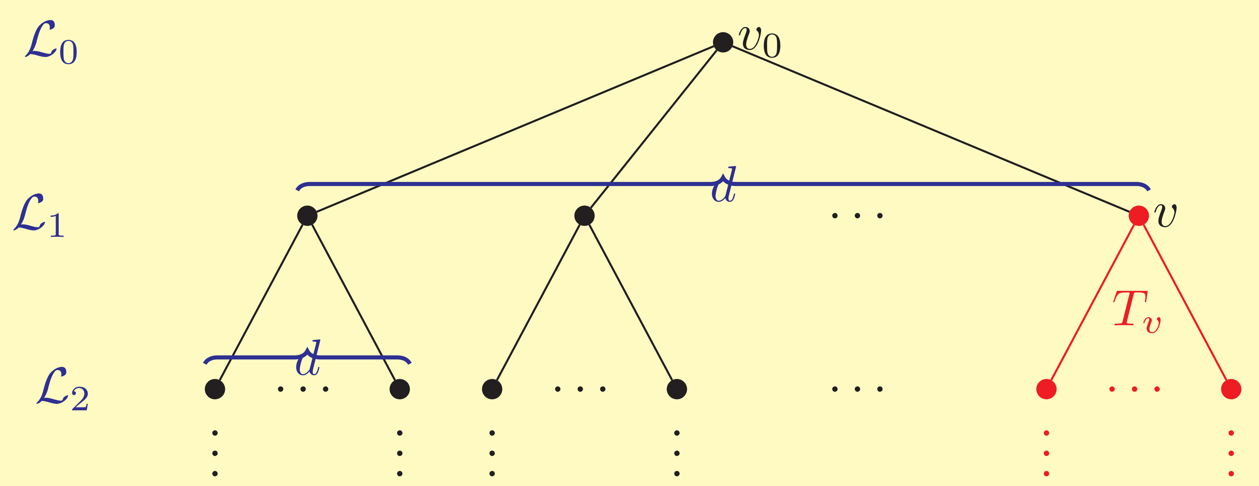
Subgroup separability A group is **subgroup separable** if all its finitely generated subgroups are closed in its profinite topology. This has connections with the generalized word problem.

Generalized word problem: For a finitely generated group and any finitely generated subgroup H, 'is there an algorithm that decides whether a word in the generators represents an element in H?'
This is solvable for subgroup separable groups with solvable word problem. [Grigorchuk, 1984]: There is an algorithm for all groups of 'spinal type', in particular for Grigorchuk and Gupta–Sidki p-groups.

Theorem (Grigorchuk and Wilson, [4])**.** *All infinite finitely generated subgroups of the Grigorchuk group are commensurable with it. The Grigorchuk group is subgroup separable.*

Regular rooted trees

T_d = d-regular rooted tree: a tree with root v_0 such that every vertex has d 'children'.



$\mathcal{L}_n$ = vertices at distance n from root.
T_v= subtree rooted at v.
Some subgroups and homomorphisms: For a group G acting on T_d (fixing v_0)

$\mathrm{St}_G(v) := \{g \in G : v^g = v\}$ is the **stabilizer** of v;

$\mathrm{St}_G(n) := \bigcap_{v\in\mathcal{L}_n} \mathrm{St}_G(v)$ is the nth **level stabilizer**.

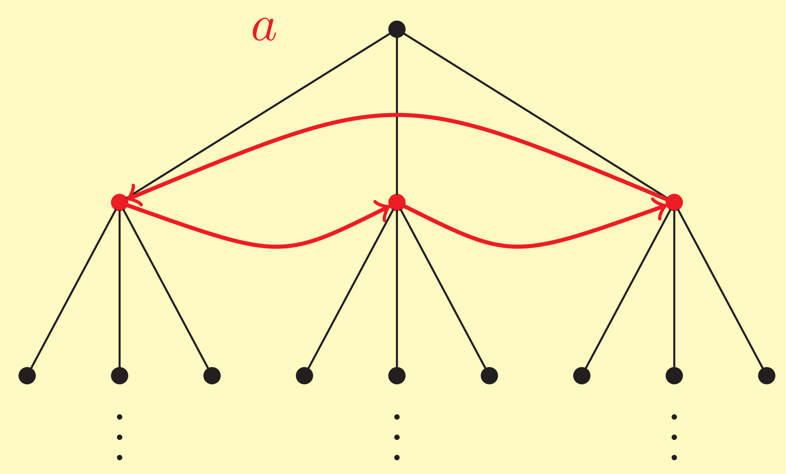
For any vertex v, for every $x \in \mathrm{St}_G(v)$ we can assign a unique $x_v \in \mathrm{Aut}\, T_v$ by restriction: $x_v := x|_{T_v}$.
If $v \in \mathcal{L}_n$, identify T_v and $T_{(n)}$ (tree rooted at level n).
Then we have a homomorphism $\varphi_v : \mathrm{St}(v) \to \mathrm{Aut}\, T_{(n)}$, $x \mapsto x_v$.

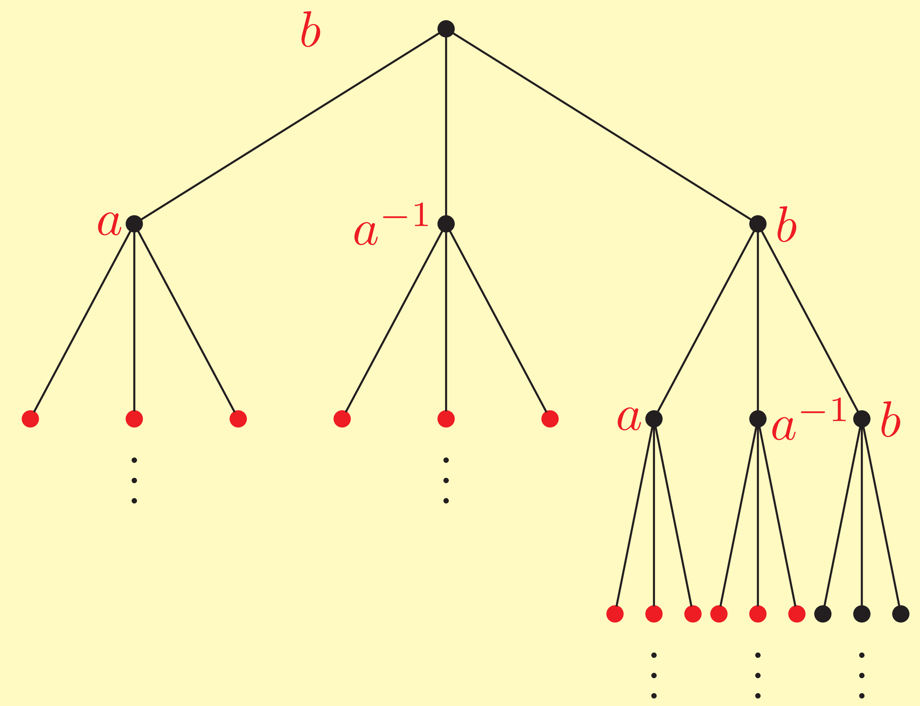
$$G_v := \varphi_v(\mathrm{St}_G(v)) \text{ is the } \textbf{projection} \text{ of } G \text{ at } v.$$

Gupta–Sidki p-groups:

Defined by Gupta and Sidki in [5]. Act on $T = T_p$ for primes $p > 2$.

$$\mathbf{G} := \langle \mathbf{a}, \mathbf{b} \rangle$$





They are residually finite, just infinite, fractal, branch, p-groups. See [1] for more.

Main results

Let G be the Gupta–Sidki 3-group.

A All infinite finitely generated subgroups of G are commensurable with G or $G \times G$.

B G is not commensurable with $G \times G$. This also holds for all Gupta–Sidki p-groups and many other branch groups (joint with J. S. Wilson, [3]).

C G is subgroup separable, therefore the generalized word problem is solvable.

Length reduction

Since G is fractal, all projections of its elements are in G. This allows the use of **length reduction arguments**, reducing word length by projecting down levels of T.
Example: the element $x = a^2b^3a^4b^2a$ for $p = 7$ can be written as

$$x = a^{-5}b^3a^5a^{-1}b^2a = (b^2, a^2, a^{-2}, 1, b^3, a^3, a^{-3}) \in \mathrm{St}_G(1).$$

For each $v \in \mathcal{L}_1$, the vth coordinate of this vector is x_v. Note that each x_v is of shorter word length than x.
Length reduction is key in the proof of Theorems A and C.

Proof ideas: key theorem

Theorem. *Let $\mathcal{X}$ be a family of subgroups of G satisfying*

1. *$1 \in \mathcal{X}, G \in \mathcal{X}$;*
2. *if $H \in \mathcal{X}$ and $H \leq_f L$ then $L \in \mathcal{X}$;*
3. *if H is finitely generated, $H \in \mathrm{St}(1)$ and $H_u \in \mathcal{X}$ for every $u \in \mathcal{L}_1$ then $H \in \mathcal{X}$.*

All finitely generated subgroups of G are in $\mathcal{X}$.

Proof by contradiction, relies on length reduction and

Lemma. *If H is a finitely generated subgroup with $H \not\leq \mathrm{St}_G(1)$ and $H_u \neq G$ for all $u \in \mathcal{L}_1$, then $H_u \leq \mathrm{St}_G(1)$.*

Find finitely generated $H \notin \mathcal{X}$ with shortest maximum length of generators. If $H \leq \mathrm{St}_G(1)$ then by 3, some projection is not in $\mathcal{X}$ and is generated by elements of $\approx$ half the length. If $H \not\leq \mathrm{St}_G(1)$, use Lemma and show all generators of H_u for $u \in \mathcal{L}_2$ are shorter.

Proof ideas: Theorems A and C

Theorem (A)**.** *All subgroups of G which are commensurable with G or $G \times G$, or finite satisfy 1–3 as in the key theorem.*

Showing 1 and 2 is easy, 3 is harder (use self-similarity and subgroup properties).

Theorem (C)**.** *All finitely generated subgroups of G all of whose finite index subgroups are closed in the profinite topology on G satisfy 1–3 as above. In particular, G is subgroup separable.*

Again, 1 and 2 are easy. To show 3 use the previous theorem and a lemma: if a group is commensurable with G or $G \times G$ then all its quotients are residually finite.

References

[1] L. Bartholdi, R. I. Grigorchuk and Z. Sunik. Branch groups. In *Handbook of Algebra*, vol. 3 (North-Holland, Amsterdam, 2003) pp. 989

[2] A. Garrido. Abstract commensurability and the Gupta–Sidki group. Transfer dissertation, University of Oxford (2012).

[3] A. Garrido and J. S. Wilson. On subgroups of finite index in branch groups. Pre-print `arXiv:1301.6419`

[4] R. I. Grigorchuk and J. S. Wilson. A structural property concerning abstract commensurability of subgroups, J. London Math. Soc. (2) **68** (2003), 671–682.

[5] N.D. Gupta and S. Sidki. On the Burnside problem for periodic groups, Math. Z. **182** (1983), 385–388.