Discontinuous actions of unit groups of orders in rational group rings $\mathbb{Q}G$

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Units in an order of $\mathbb{Q}G$

Motivation

Definition 1. Let G be a group and R a ring. The **group ring** RG is defined as the set of all linear combinations of the form $\sum_{g\in G} a_g g$ with $a_g \in R$ and only a finite number of a_g are non zero. The sum of two elements is defined componentwise:

$$(\sum_{g\in G} a_g g) + (\sum_{g\in G} b_g g) = \sum_{g\in G}(a_g + b_g)g$$

and the product is given by

$$(\sum_{g\in G} a_g g)(\sum_{g\in G} b_g g) = \sum_{g,h\in G} a_g b_h gh.$$

Definition 2. Let A be a $\mathbb{Q}$-algebra. A subring R of A containing its unity is called an **order** in A if R is finitely generated as a $\mathbb{Z}$-module and $\mathbb{Q}R = A$.

Example 3. $\mathbb{Z}$ is an order in $\mathbb{Q}$ and $\mathbb{Z}G$ is an order in $\mathbb{Q}G$.

Open problem: Finding a presentation of $\mathcal{U}(\Gamma)$, where Γ is an order in $\mathbb{Q}G$, for G a finite group, in particular describing $\mathcal{U}(\mathbb{Z}G)$.

General Approach

By the Wedderburn-Artin Theorem,

$$\mathbb{Q}G = \prod_{i=1}^{n} M_{n_i}(D_i).$$

- let $\mathcal{O}_i$ be an order in D_i,
- set $\mathcal{O} = \prod_{i=1}^{n} M_{n_i}(\mathcal{O}_i)$,
- $\mathcal{U}(\mathcal{O})$ and $\mathcal{U}(\mathbb{Z}G)$ (resp. $\mathcal{U}(\mathcal{O}')$ for $\mathcal{O}'$ an order in $\mathbb{Q}G$) are commensurable,

Theorem 4 (Commensurability of two orders in a $\mathbb{Q}$-algebra)**.** *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two orders in a $\mathbb{Q}$-algebra. Then there exists an order $\mathcal{O}$ such that $[\mathcal{O}_i : \mathcal{O}] < \infty$ for $i = 1, 2$. $\mathcal{O}_1$ and $\mathcal{O}_2$ are said to be commensurable.*

- $\mathcal{U}(\mathcal{O}) = \prod_{i=1}^{n} \mathrm{GL}_{n_i}(\mathcal{O}_i)$,
- for every $1 \le i \le n$, $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ is commensurable with $\mathcal{U}(\mathcal{Z}(\mathcal{O}_i)) \times \mathrm{SL}_{n_i}(\mathcal{O}_i)$.

Hence the original problem is reduced to the one of finding a presentation of $\mathrm{SL}_{n_i}(\mathcal{O}_i)$.

For many finite groups G:

- specific finite set B of generators of a subgroup of finite index in $\mathcal{U}(\mathbb{Z}G)$ given in a purely algebraic way.

Theorem 5 (Bass,Vaseršteĭn and Venkataramana and Kleinert)**.** *Let $\mathcal{O}$ be a maximal order in a finite dimensional rational division algebra D with center K (a number field). If*

- *$n \ge 3$ or*
- *$n = 2$ and D is different from $\mathbb{Q}$, a quadratic imaginary extension of $\mathbb{Q}$ and a totally definite quaternion algebra with center $\mathbb{Q}$,*

then a finite group B of generators may be given for $\mathrm{SL}_{n_i}(\mathcal{O})$. If

- *$n = 1$ and D is either commutative or a totally definite quaternion algebra over $\mathbb{Q}$,*

then $\mathcal{U}(\mathcal{O})$ is finite.

We call the group algebra $\mathbb{Q}G$ of **exceptional type** if it has a simple component which is equal to

- a non-commutative division ring different from a totally definite quaternion algebra,
- $M_2(D)$ with D equal to $\mathbb{Q}$ or $\mathbb{Q}(\sqrt{-d})$ with $d > 0$,
- a 2 by 2 matrix ring over a totally definite rational quaternion algebra $\mathcal{H}(a, b, \mathbb{Q})$.

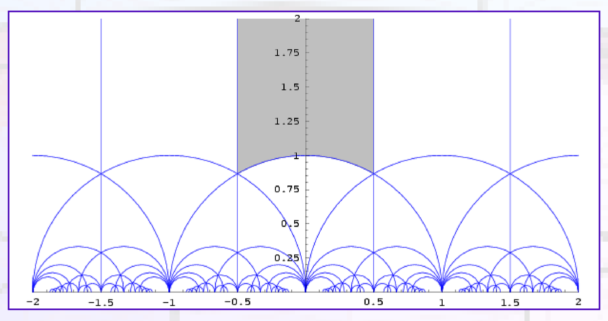
→ Idea for these cases: **hyperbolic geometry**

The Poincaré Theorem

Definition 6. A closed subset $\mathcal{F} \subseteq X$, with X a metric space, is called a **fundamental domain** of the discontinuous group $\Gamma < \mathrm{Iso}(X)$ if the following conditions are satisfied:

- the set $\mathcal{F}$ is closed and connected in X,
- the members of $\{g(\mathcal{F}^\circ) \mid g \in \Gamma\}$ are mutually disjoint, and
- $X = \bigcup_{g\in\Gamma} g(\mathcal{F})$.

Example 7.



Fundamental domain of $\mathrm{SL}_2(\mathbb{Z})$

Theorem 8 (Poincaré)**.** *Let $\mathcal{F}$ be a convex fundamental domain, which is a polyhedron, for a discrete group Γ of $\mathbb{H}^n$. Then Γ is generated by*

$$\Phi = \{g \in \Gamma \mid \mathcal{F} \cap g(\mathcal{F}) \text{ is a side of } \mathcal{F}\}.$$

The Poincaré method gives also a method for finding relations in the presentation of the group, based on the sides and the edges of the fundamental polyhedron.

Link to $\mathcal{U}(\mathbb{Z}G)$

Components of the algebra $\mathbb{Q}G$ of exceptional type:

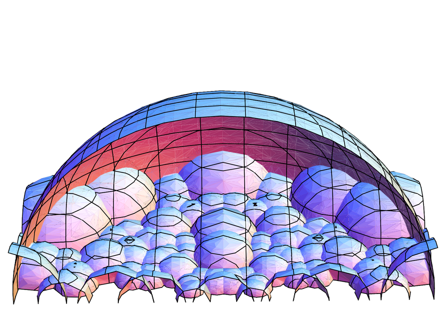
- $\mathcal{H}(a, b, \mathbb{Q}(\sqrt{-d}))$ for d a square-free positive integer,
- $M_2(\mathbb{Q}(\sqrt{-d}))$ for d a square-free positive integer.

Unit groups of orders in these algebras:

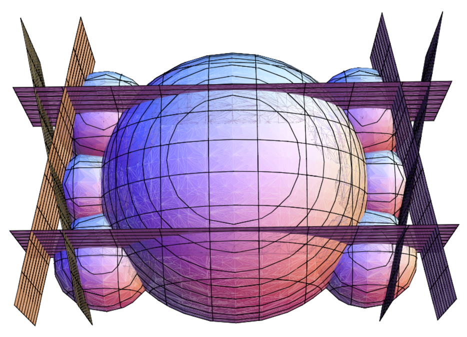
- act as isometries via Möbius action on $\mathbb{H}^3$,
- discrete in $\mathrm{SL}_2(\mathbb{C})$ and hence discontinuous action on $\mathbb{H}^3$.

→ generators via the Poincaré Theorem

Example 9.



$\mathrm{SL}_1\left(\mathcal{H}\left(-1, -1, \mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]\right)\right)$



$\mathrm{PSL}_2\left(\mathbb{Z}\left[\frac{1+\sqrt{-23}}{2}\right]\right)$

Example 10. Background: $\mathrm{SL}_1(\mathcal{H}(2, 5, \mathbb{Z}[i]))$

Summary:

- Problem of purely algebraic nature: group of units in an order in a rational group ring,
- Translate problem to hyperbolic geometry: embed group in $\mathrm{SL}_2(\mathbb{C})$ and let it act on hyperbolic space,
- Apply the Poincaré Theorem to get a presentation of the group,
- Re-translate back to groups rings.

Done in [4].

Product of Hyperbolic Spaces

More difficult context: D a classical quaternion algebra over $\mathbb{Z}[\xi_n]$ where ξ_n is a n-th primitve root of unity.

Example 11. The easiest case: $\mathbb{Q}(Q_8 \times C_7)$.

$$\mathcal{U}(\mathbb{Z}(Q_8 \times C_7)) \rightsquigarrow \mathcal{U}(\mathcal{H}(-1, -1, \mathbb{Z}[\xi_7])) \hookrightarrow \mathrm{SL}_2(\mathbb{Z}[\xi_7]).$$

- $\mathrm{SL}_2(\mathbb{Z}[\xi_7])$ not discrete in $\mathrm{SL}_2(\mathbb{C})$,
- discreteness in $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$,

→ hence discontinuous action on $\mathbb{H}^3 \times \mathbb{H}^3 \times \mathbb{H}^3$.

Question: Does the Poincaré Theorem still work in this case?

Definition 12. Let $K = \mathbb{Q}(\sqrt{d})$ with d a square-free positive integer and let

$$\mathcal{O} = \mathbb{Z}\left[\frac{1+\sqrt{d}}{d_0}\right], \text{ with } d_0 = \begin{cases} 1, \text{ if } d \not\equiv 1 \mod 4; \\ 2, \text{ if } d \equiv 1 \mod 4. \end{cases}$$

The Hilbert Modular Group $\mathcal{H}$ is the subgroup of $\mathrm{GL}_2(\mathcal{O})$ consisting of matrices P with $det(P) \gg 0$.

Test case: Hilbert Modular Group $\mathcal{H}$, with K such that $\mathcal{O}$ PID.

Lemma 13. *An embedding of the group $\mathcal{H}$ is discrete in $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ and has thus a discontinuous action on $\mathbb{H}^2 \times \mathbb{H}^2$.*

- ∗ Is it possible to find a fundamental domain?
  - Yes, but no longer a polyhedra.
- ∗ Concerning generators: what is a side of the fundamental domain?
  - Find an adequate definition of a side.
- ∗ Concerning relations: what is an edge of the fundamental domain?
  - Find an adequate definition of an edge.

Done in [5].

Further Work

- generalize the theory to $K = \mathbb{Q}(\sqrt{d})$ with $\mathcal{O}$ not PID,
- generalize the theory to more copies of hyperbolic (3−)space,
- attack the concrete problem of $\mathcal{U}(\mathbb{Z}(Q_8 \times C_7))$.

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