Locally compact groups for which the topology is essentially unique

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Background

Locally compact groups

By a locally compact group we mean a topological group which is Hausdorff and locally compact. We shall be mostly interested in locally compact groups which are almost simple, in the sense that every normal subgroup is either equal to the entire group, or finite and central. Some examples of this are the group of k-rational points of an absolutely almost simple algebraic group defined over a non-archimedean local field k, or the full automorphism group of a locally finite biregular tree.

The problem

Try to determine which locally compact groups in this class have the property that they admit just one locally compact σ-compact Hausdorff group topology.

The case of simple Lie groups

By a simple Lie group we shall mean a connected centreless Lie group with a simple Lie algebra. By an absolutely simple Lie group we shall mean a connected centreless Lie group such that if $\mathfrak{s}$ is the Lie algebra then $\mathfrak{s} \otimes_{\mathbb{R}} \mathbb{C}$ is a simple Lie algebra over $\mathbb{C}$.

Theorem. *Suppose that S is an absolutely simple Lie group and that $G \subseteq \mathit{AutLie}(S)$ is an open subgroup. Let Γ be a locally compact and σ - compact Hausdorff topological group and assume that $\varphi : \Gamma \to G$ is an abstract surjective group isomorphism. Then φ is a homeomorphism. [1]*

Some important ingredients in the proof

A key ingredient in the proof is the following theorem.

Theorem. *Let Γ be a locally compact and σ-compact Hausdorff topological group. Let S be a simple Lie group, let G be an open subgroup of $\mathit{AutLie}(S)$ and suppose that $\varphi : \Gamma \to G$ is an abstract surjective group homomorphism. Suppose that there is a compact subset $C \subseteq G$ which contains a nonconstant smooth curve and whose φ-preimage $\varphi^{-1}(C)$ is σ-compact. Then φ is continuous and open.*

The basic idea is to choose the set C in such a way that $\varphi^{-1}(C)$ will be σ-compact regardless of what the topology is on Γ, so long as it is compatible with the group operations of Γ. This will be the case if C is the centraliser of a set, for example, or a conjugacy class.

The case of an algebraic group over a non-archimedean local field

In generalising Linus Kramer's argument to absolutely almost simple algebraic groups defined over a non-archimedean local field k, a similar line of reasoning is used. But this time the requirement is that C contain the k-rational points of a constructible set of dimension greater than zero, rather than a smooth curve. This is discussed in [2].

The case of the full automorphism group of a locally finite biregular tree

Here the key trick is to find a candidate for C whose preimage will be σ-compact in any group topology on Γ, and such that finitely many conjugates of C generate all of a vertex stabiliser.

Conclusions

These preliminary investigations give some preliminary evidence that this property of the "rigidity of the group topology" could hold for a very large class of locally compact groups. Further investigation of how these techniques can be generalised to other locally compact groups should prove fruitful. One conjecture which it would be particularly interesting to verify is the following: Suppose that a closed subgroup G of the full automorphism group of a locally finite biregular tree acts 2-transitively on the ends of the tree. Then G admits just one locally compact σ-compact Hausdorff group topology. This would have the consequence that the structure of the tree could be recovered from the group. This would have a bearing on related questions about locally finite Euclidean buildings.

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Bibliography

[1] Linus Kramer, *The topology of a simple Lie group is essentially unique*, Advances in Mathematics (2011), no. 228, 2623–2633.

[2] Rupert McCallum, *Rigidity of the group topology for p-adic semisimple groups and automorphism groups of trees*.