Conjugacy classes in locally compact, totally disconnected, and second countable groups

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Definitions and examples

- A topological group is *locally compact, totally disconnected, and second countable* if the underlying topological space has these properties.
- **Examples**
  - Countably based profinite groups: e.g. $\mathbb{Z}_3^{\mathbb{N}}$ with the product topology
  - *Isom*(Γ) with the topology of pointwise convergence for Γ a countably infinite, connected, and locally finite graph
- A set is *comeagre* in a space if it has a subset which is a dense G_δ in the space.

Facts

(i) There are many non-trivial, non-locally compact, totally disconnected, and second countable groups with a comeagre conjugacy class. E.g. *Sym*$(\mathbb{N})$, *Aut*$(\mathbb{Q}, <)$ (folklore, Truss)

(ii) A non-trivial compact group G cannot have a dense conjugacy class.

(iii) There is an infinite, profinite, and second countable group with a non-meagre conjugacy class. (Rosendal)

(iv) There is a non-trivial, locally compact, totally disconnected, and second countable group with a dense conjugacy class. (Akin, Glasner, Weiss [1])

Motivation

The above facts show that compact groups are "too small" for even a dense conjugacy class. While, non-locally compact groups are "large enough" to admit a comeagre conjugacy class. Naturally, one asks where locally compact groups fall. I.e. are they still "too small" for a comeagre conjugacy class or are they "large enough" to admit a comeagre conjugacy class? A priori, one may guess that locally compact groups are "too small". However, items **(iii)** and **(iv)** indicate the opposite.

Question (Kechris, Rosendal [2])

Can a non-trivial, locally compact, totally disconnected, and second countable group have a comeagre conjugacy class?

Background on Profinite groups

Let U be a profinite group.

- The *Frattini* subgroup of U, $\Phi(U)$, is the intersection of all maximal, proper open subgroups.
- $\Phi(U)$ is the collection of non-generators of U. In particular, if $U = H\Phi(U)$ for $H \leq_c U$, then $U = H$.
- If U is pro-p, then $\Phi(U) = U^p[U, U]$ where $[U, U]$ is the closure of the commutator subgroup.

Proofs of these and a nice introduction to profinite group theory may be found in [4].

Profinite Structure Theorems

- (Zel'manov [6]) Every torsion pro-p group is locally finite.
- (Wilson, [5]) Let U be a compact Hausdorff torsion group. Then U has a finite series

$$\{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U$$

of closed characteristic subgroups in which each factor U_i/U_{i-1} is either **(1)** pro-p for some prime p or **(2)** isomorphic to a Cartesian product of isomorphic finite simple groups.

Key Lemma

Let U be a profinite group with normalized Haar measure μ and h^U denote the conjugacy class of h in U. If $h \in U$ is such that $\mu(h^U) > 0$, then $|C_U(h)| < \infty$.

Proof.

Take $N \trianglelefteq U$ an open normal subgroup.

- Take a transversal $h, k_1hk_1^{-1}, .., k_nhk_n^{-1}$ for $(hN)^{U/N}$ in U/N. Then $h^U \subseteq hN \cup \ldots \cup k_nhk_n^{-1}N$.
- So, $\mu(h^U) \leq |(hN)^{U/N}|\mu(N)$. And,

$$\mu(N)|U/N : C_{U/N}(hN)| = \frac{\mu(N)|U/N|}{|C_{U/N}(hN)|} = \frac{1}{|C_{U/N}(hN)|}$$

- Whence, $|C_{U/N}(hN)| \leq \frac{1}{\mu(h^U)}$, and it follows $|C_U(h)| \leq \frac{1}{\mu(h^U)}$.

□

Lemmas

1. Let G be totally disconnected, locally compact, and second countable. If g^G is non Haar null, then g^U is non Haar null for all compact, open subgroups U of G.
2. Let G be as above. If $g \in P_1(G) := \{g \in G : cl(\langle g \rangle) \text{ is compact}\}$ and $\mu(g^G) > 0$, then g is a torsion element. **proof:**
   - Find W compact, open, and containing g.
   - g^W is non Haar null by **lemma (1)**.
   - By the key lemma, g is torsion. □
3. A torsion pro-p group with a non Haar null conjugacy class is finite. **proof:**
   - Suppose U is a torsion, pro-p group with a non-null conjugacy class h^U.
   - $h^{-1}h^U \subseteq [U, U]$ is non-null where $[U, U]$ is the closure of the commutator subgroup.
   - $[U, U]$ is open by the Steinhaus-Weil theorem. For a nice proof see [3].
   - Since U is pro-p, $[U, U] \subseteq \Phi(U)$ with $\Phi(U)$ the Frattini subgroup of U.
   - Letting $k_1, .., k_n$ be coset representatives for $\Phi(U)$ in U, $U = cl(\langle k_1, .., k_n \rangle)\Phi(U)$. So U is finitely generated and, therefore, finite by Zel'manov's theorem. □

Theorem [W.]

If G is a non-trivial, locally compact, totally disconnected, and second countable group and g^G is a dense conjugacy class, then g^G is Haar null.

Proof.

- Suppose $g \in G$ is such that g^G is dense and non-null for contradiction.
- Wlog $g \in P_1(G)$ and is thus torsion by **lemma (2)**. It follows G has exponent $|g|$.
- Take U a compact open subgroup. By Wilson's theorem, we have

$$\{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U$$

- Let $k < n$ be greatest such that U_k is not open in U.
- U_{k+1}/U_k then has a non-null conjugacy class. U_{k+1}/U_k cannot be pro-p by **lemma (3)**. If a product, the product must be finite since there is an element with finite centralizer.
- This contradicts the choice of k.

□

Corollary

A non-trivial, locally compact, totally disconnected, and second countable group cannot have a comeagre conjugacy class.

Proof.

- Suppose g^G is comeagre for contradiction.
- Since G is K_σ, we may take $G = \bigcup_{i\in\omega} K_i$ with K_i compact.
- $g^G = \bigcup_{i\in\omega} g^{K_i}$; so, some g^{K_i} is non meagre by the Baire category theorem.
- g^{K_i} is closed and so contains an open set O.
- Thus $g^G = O^G$ is open and $\mu(g^G) > 0$.
- This contradicts the theorem.

□

Remarks

- The corollary answers Kechris and Rosendal's question for all locally compact, second countable groups by an unpublished result of Professor K.H. Hofmann.
- A pre-print of the result above along with Hofmann's result will be posted to the arXiv in the near future. If you would like a copy, feel free to email me: pwesol3@uic.edu

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