The scale on totally disconnected, locally compact groups

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April 3rd 2013

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The scale function

G – a totally disconnected, locally compact group
α – a bicontinuous automorphism of G
$\mathcal{B}(G)$ – the set of compact, open subgroups of G.

Definition

Let $\alpha \in \mathrm{Aut}(G)$. The *scale of α* is the positive integer

$$s(\alpha) := \min \{[\alpha(V) : \alpha(V) \cap V] \mid V \in \mathcal{B}(G)\}.$$

The compact open subgroup V of G is *minimizing for α* if the minimum is attained at V.

$s : \mathrm{Aut}(G) \to \mathbb{Z}^+$ is the *scale function*.
Restricting to inner automorphisms yields $s : G \to \mathbb{Z}^+$, where $s(x) = s(\alpha_x)$ and $\alpha_x(y) = xyx^{-1}$.

Properties of the scale

Theorem (Properties of the scale)

The function $s : \mathrm{Aut}(G) \to \mathbb{Z}^+$ *satisfies:*

1. $s(\alpha) = 1 = s(\alpha^{-1})$ *if and only if there is* $V \in \mathcal{B}(G)$ *such that* $\alpha(V) = V$ *(in this case* α *is uniscalar);*
2. $s(\alpha^n) = s(\alpha)^n$ *for every* $n \in \mathbb{N}$*; and*
3. $\triangle(\alpha) = s(\alpha)/s(\alpha^{-1})$, *where* $\triangle : \mathrm{Aut}(G) \to \mathbb{R}^+$ *is the modular function.*
4. *The function* $s : G \mapsto \mathbb{Z}^+$ *is continuous for the group topology on* G *and the discrete topology on* $\mathbb{Z}^+$.

Proposition

The set $P(G) = \left\{ x \in G \mid \overline{\langle x \rangle} \text{ is compact} \right\}$ is closed.

Minimizing subgroups are tidy

Theorem (Characterization of minimizing subgroups)

Let $\alpha \in \mathrm{Aut}(G)$ *and* $V < G$ *be compact and open. Define*

$$V_+ := \bigcap_{k \geq 0} \alpha^k(V) \text{ and } V_- := \bigcap_{k \geq 0} \alpha^{-k}(V).$$

Then V *is minimizing for* α *if and only if*

TA $V = V_+ V_-$; *and*

TB $V_{++} := \bigcup_{k \geq 0} \alpha^k(V_+)$ *is closed.*

If V *is minimizing, then* $s(\alpha) = [\alpha(V_+) : V_+]$.

Tidy subgroups

Definition

A subgroup $V \in \mathcal{B}(G)$ satisfying **TA** and **TB** is *tidy for* α.

In examples, identifying a subgroup tidy for α corresponds to putting α into a relevant canonical form.

- When G is a p-adic Lie group, put $D(\alpha)$ in Jordan canonical form to compute $s(\alpha)$ and find tidy subgroups.
- When G is the automorphism of a tree and $\alpha = \alpha_x$ for $x \in G$, all tidy subgroups lie along the axis of x and the scale measures the translation distance of x.

Flat groups of automorphisms

Theorem

Let $\mathcal{H}$ be a finitely generated group of automorphisms of G and suppose that V is tidy for $\mathcal{H}$. Then $\mathcal{H}_1 = \{\alpha \in \mathcal{H} \mid \alpha(V) = V\}$ is a normal subgroup of $\mathcal{H}$ and there is $r \in \mathbb{N}$ such that

$$\mathcal{H}/\mathcal{H}_1 \cong \mathbb{Z}^r.$$

1. *There is $q \in \mathbb{N}$ such that*

$$V = V_0 V_1 \dots V_q,$$

where for every $\alpha \in \mathcal{H}$: $\alpha(V_0) = V_0$ and for every $j \in \{1, 2, \dots, q\}$ either $\alpha(V_j) \leq V_j$ or $\alpha(V_j) \geq V_j$.

2. For each $j \in \{1, 2, \ldots, q\}$ there is a homomorphism $\rho_j : \mathcal{H} \to \mathbb{Z}$ and a positive integer s_j such that
$$[\alpha(V_j) : V_j] = s_j^{\rho_j(\alpha)}.$$
3. For each $j \in \{1, 2, \ldots, q\}$,
$$\widetilde{V}_j := \bigcup_{\alpha \in \mathcal{H}} \alpha(V_j)$$
is a closed subgroup of G.
4. The natural numbers r and q, the homomorphisms $\rho_j : \mathcal{H} \to \mathbb{Z}$ and positive integers s_j are independent of the subgroup V tidy for $\mathcal{H}$.

Flat groups of automorphisms

- The numbers $s_j^{\rho_j(\alpha)}$ are analogues of absolute values of eigenvalues for α.
- The subgroups $\bigcup_{\alpha\in\mathcal{H}} \alpha(V_j)$ are the analogues of common eigenspaces for the automorphisms in $\mathcal{H}$.

Definition

A group $\mathcal{H} \leq \mathrm{Aut}(G)$ for which there is a common tidy subgroup is called *flat*.

Example

$G = SL(n, \mathbb{Q}_p)$, $H = \{$ diagonal matrices in $GL(n, \mathbb{Q}_p)\}$ and $\alpha_h(x) = hxh^{-1}$. Then:

- $r = n - 1$;
- ρ_j are roots of H; and
- $\widetilde{V}_j$ are root subgroups of G.

Flat groups of automorphisms

Theorem

The subgroup $\mathcal{H} \leq \mathrm{Aut}(G)$ *is:*

1. *flat if it is finitely generated and nilpotent; and*
2. *virtually flat if it is polycyclic.*

Remark

A single linear transformation T has a spectrum of eigenvalues which is an artefact of the algebra generated by T.
There is no similar method to generate a large number of automorphisms commuting with a single $\alpha \in \mathrm{Aut}(G)$.

Contraction groups

Definition

The *contraction group* of the automorphism α is

$$\mathrm{con}(\alpha) = \{x \in G \mid \alpha^n(x) \to 1 \text{ as } n \to \infty\}\,.$$

Theorem (U. Baumgartner & W., Jaworski)

Let V be tidy for α. Then $V_{--} = V_0 \cdot \mathrm{con}(\alpha)$. Hence

$$s(\alpha^{-1}) > 1 \implies \mathrm{con}(\alpha) \textit{ unbounded } \implies \mathrm{con}(\alpha) \neq \{1\}.$$

Theorem (U. Baumgartner & W.)

$$\overline{\mathrm{con}(\alpha)} = \bigcap \{V_{--} \mid V \textit{ is tidy for } \alpha\}\,.$$

Contraction groups and the scale

The closure of con(α) carries the scale value:

$$s(\alpha^{-1}) = s(\alpha^{-1}|_{\overline{\mathrm{con}(\alpha)}}).$$

Contraction subgroups of Lie groups are automatically closed but that it not so for automorphisms of general t.d.l.c. groups. Closedness of contraction groups is quite restrictive for t.d.l.c. groups. H. Glöckner will speak about that.

The 'obstruction' to con(α) being closed is as follows.

The nub and closedness of $\mathrm{con}(\alpha)$

Definition

The *nub* of the automorphism α is

$$\mathrm{nub}(\alpha) = \bigcap \{ V \mid V \text{ is tidy for } \alpha \} .$$

Theorem (U. Baumgartner & W.)

$\mathrm{con}(\alpha)$ *is closed if and only if* $\mathrm{nub}(\alpha)$ *is trivial.*

Many of the following results for the nub have counterparts in work in topological dynamics by B. Kitchens and K. Schmidt.

Further descriptions and properties of the nub

Theorem

The nub of α is:

- *the largest subgroup of G on which α acts ergodically;*
- *equal to the closures of* $\mathrm{bco}(\alpha)$ *and* $\mathrm{bco}(\alpha^{-1})$*, where*

$$\mathrm{bco}(\alpha) := \{x \in \mathrm{con}(\alpha) \mid \{\alpha^n(x)\}_{n\in\mathbb{Z}} \text{ is bounded}\}.$$

Remarks

- Any l.c. group on which an automorphism acts ergodically must be compact. (Conjectured by Halmos in 1950's.)
- $\mathrm{bco}(\alpha) \cap \mathrm{bco}(\alpha^{-1})$ is the *homoclinic* subgroup for α. This intersection need not be dense in $\mathrm{nub}(\alpha)$.

The structure of the nub

Let $G = F^{\mathbb{Z}}$, with F a finite group. Then G is a compact group. Let α be the *shift*, *i.e.*, $\alpha(f)_n = f_{n+1}$ for $f \in F^{\mathbb{Z}}$.
The the homoclinic subgroup of G is the set of sequences with finite support, which is dense in G. Hence $\text{nub}(\alpha) = G$.

The following shows how $\text{nub}(\alpha)$ may be built from groups of this type.

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Definition

Let G be a t.d. compact group and $\alpha \in \mathrm{Aut}(G)$. Then (G, α) has *finite depth* if there is an open subgroup V with

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(V) = \{1\}.$$

Such an α is called *expansive* in topological dynamics.

Proposition

Let G be a t.d. compact group and $\alpha \in \mathrm{Aut}(G)$. Then

$$(G, \alpha) = \varprojlim (G_\iota, \alpha_\iota)$$

where each (G_ι, α_ι) has finite depth.

The structure of the nub 3

Theorem (Jordan-Holder for the nub)

Let (G, α) *have finite depth and suppose that* $\mathrm{nub}(\alpha) = G$*, i.e.,* α *acts ergodically on* G*. Then* (G, α) *has a composition series*

$$G_0 = \{1\} \leq G_1 \leq \dots G_{r-1} \leq G_r = G,$$

where $G_k \triangleleft G_{k+1}$ *for every* $k \in \{0, 1, \dots, r-1\}$ *and* $(G_{k+1}/G_k, \alpha_{G_{k+1}}^{G_{k+1}/G_k})$ *is isomorphic to a shift* $F_k^{\mathbb{Z}}$ *with* F_k *a finite simple group.*

A consequence of the structure theorem

Corollary

Let G be a t.d. compact group and $\alpha \in \mathrm{Aut}(G)$ satisfy that $\mathrm{nub}(\alpha) = G$. Then the map

$$x \mapsto \alpha(x)x^{-1}, \quad G \to G,$$

is surjective.