Quasi-Clifford Algebras, their Lie and Jordan Algebras, and Quadratic Forms over $\mathbb{F}_2$

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Abstract

Let R be a commutative and associative ring containing distinct elements 1 and -1. Let $\Gamma = (\mathcal{V}, \mathcal{E}, \lambda)$ be a labeled graph, with vertex set $\mathcal{V}$, edge set $\mathcal{E}$ and labeling of the vertices $\lambda : \mathcal{V} \to R^*$ which maps $v \in \mathcal{V}$ to invertible elements $\lambda(v) \in R^*$. Then we consider the associative algebra $\mathfrak{C}(\Gamma)$ with identity element $\mathbf{1}$ generated by the elements of $\mathcal{V}$ such that for all $v, w \in \mathcal{V}$ we have

$$\begin{array}{lll} v^2 & = \lambda(v)\mathbf{1}, & \\ vw + wv & = 0 & \text{if } \{v,w\} \in \mathcal{E}, \\ vw - wv & = 0 & \text{if } \{v,w\} \notin \mathcal{E}. \end{array}$$

If Γ is the complete graph, $\mathfrak{C}(\Gamma)$ is a Clifford algebra, otherwise it is a so-called quasi-Clifford algebra, as in [2].

We describe this algebra as a twisted group algebra with the help of a vector space V over the field $\mathbb{F}_2$ equipped with a bilinear form g. See also [1]. Using this description, we determine the isomorphism type of $\mathfrak{C}(\Gamma)$ for several interesting graphs Γ.

As the algebra $\mathfrak{C}(\Gamma)$ is associative, we can also consider the corresponding Lie algebra with Lie bracket $[\cdot,\cdot]$ and Jordan algebra with multiplication $\circ$, as well as some of their subalgebras. We find that the elements $v, w \in \mathcal{V}$ satisfy the following relations

$$\begin{array}{lll} [v,w] & = 0 & \text{if } \{v,w\} \notin \mathcal{E}, \\ [v,[v,w]] & = \lambda(v)w & \text{if } \{v,w\} \in \mathcal{E}. \end{array}$$

and

$$\begin{array}{lll} v \circ v & = \lambda(v)\mathbf{1} & \\ v \circ w & = 0 & \text{if } \{v,w\} \in \mathcal{E}, \\ v \circ (v \circ w) & = \lambda(v)w & \text{if } \{v,w\} \notin \mathcal{E}. \end{array}$$

We provide characterizations of both the Lie and Jordan algebras generated by the elements in $\mathcal{V}$, as algebras defined by these relations.

In case R is a field of characteristic 0, we can identify these Lie algebras with quotients of the compact subalgebras of Kac-Moody Lie algebras and prove that they admit a so-called generalized spin representation, generalizing the work of Hainke, Köhl and Levy [3].

Keywords: Clifford algebra, Lie algebra, Jordan algebra, quadratic forms
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References

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