Geometries from structurable algebras and inner ideals

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Opening new horizons

Content

Credits

- Most work is joint with Tom De Medts (Ghent University), my supervisor.
- Work on extremal geometries is joint with Hans Cuypers (Eindhoven University of Technology).

Overview/motivation

- Structurable algebras are a class of non-associative, non-commutative algebras.
- Boelaert, De Medts, Stavrova, '19: Connection between structurable division algebras and so-called Moufang sets ("geometries of rank 1").
- Cohen, Ivanyos, '06: 1-dimensional inner ideals in a Lie algebra are related to point-line geometries ("extremal geometries").
- Tits-Kantor-Koecher-procedure, '78: We can associate a Lie algebra to a structurable algebra.
- Question: Constructions for higher rank geometries?
- Question: Relation with extremal geometries?
- Ingredient: Inner ideals!

Structurable algebras: what are these?

Assumption
All algebras are finite-dimensional and defined over a field k of characteristic different from 2 and 3! Moreover, we do not assume them to be associative nor commutative.

Definition.
Consider an algebra $\mathcal{A}$ with involution. (I.e., $\overline{xy} = \overline{y}.\overline{x}$)
Set

$$V_{x,y}z = (x\overline{y})z + (z\overline{y})x - (z\overline{x})y.$$

Then we call $\mathcal{A}$ structurable if

$$[V_{x,y}, V_{a,b}] = V_{V_{x,y}a,b} - V_{a,V_{y,x}b}.$$

Hence $\langle V_{x,y} \mid x, y \in \mathcal{A}\rangle \leq \mathrm{End}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{A})$.

Jordan algebras

- Assume that the involution is the identity.
- Then $xy = \overline{xy} = \overline{y} \cdot \overline{x} = yx$. I.e., $\mathcal{A}$ is commutative.
- Moreover, one can show that $\mathcal{A}$ is a Jordan algebra. (So satisfying $(xy)x^2 = x(yx^2)$)
- Example: A associative, then $A^+ = (A, \circ)$ Jordan, with $a \circ b = (ab + ba)/2$.
- Example: Jordan algebra associated to a quadratic form.
- Exceptional example: 27-dimensional Albert algebra, constructed from octonions.

Classification

Theorem.
If $\mathcal{A}$ is a central simple structurable algebra, it is isomorphic to (at least) one of the following:

- Central simple associative algebra with involution.
- Central simple Jordan algebra.
- Hermitian type.
- Central simple structurable algebra of skew-dimension one.
- Forms of tensor product of two composition algebras.
- Smirnov algebra.

Classification

The skew-dimension is $\dim(\mathcal{S})$, with

$$\mathcal{S} := \{a \in \mathcal{A} \mid \overline{a} = -a\}.$$

Classification due to:

- Characteristic 0: Allison, '79.
- Characteristic $\neq 2, 3, 5$: Smirnov, '91.
- Characteristic $= 5$: Boelaert, De Medts, Stavrova, '19, together with separate article of Stavrova, '19.

Tensor product of composition algebras

Definition.
If C_i is a composition algebra over k with involution σ_i, for $i = 1, 2$, then the k-algebra $C_1 \otimes_k C_2$, together with the involution

$$- = \sigma := \sigma_1 \otimes \sigma_2$$

is a structurable algebra.

- One notes $\mathcal{S} = \{s_1 \otimes 1 + 1 \otimes s_2 \mid s_i \in S_i\}$, with S_i the set of skew elements w.r.t. the involution σ_i. In particular, it is of dimension 7, 8, 10 or 14 if C_1 is octonion.
- There only exist forms if $\dim(C_1) = \dim(C_2)$.

Skew-dimension one

Let J be a Jordan algebra, $T : J \times J \to k$ be a symmetric bilinear form, $\times : J \times J \to J$ be a symmetric bilinear map, and $N : J \to k$ be a cubic form such that one of the following holds:

- J is a cubic Jordan algebra with a non-degenerate admissible form N, with basepoint 1, trace form T, and Freudenthal cross product $\times$.
- J is a Jordan algebra of a non-degenerate quadratic form q with basepoint 1, and T is the linearization of q. In this case, N and $\times$ are the zero maps.
- $J = 0$, and the maps N, T and $\times$ are the zero maps. In this case, J is not unital.

Skew-dimension one

Let

$$\mathcal{A} = \left\{ \begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \mid k_1, k_2 \in k, j_1, j_2 \in J \right\},$$

and define the multiplication and the involution by

$$\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix} \begin{pmatrix} k_1' & j_1' \\ j_2' & k_2' \end{pmatrix} = \begin{pmatrix} k_1 k_1' + T(j_1, j_2') & k_1 j_1' + k_2' j_1 + j_2 \times j_2' \\ k_1' j_2 + k_2 j_2' + j_1 \times j_1' & k_2 k_2' + T(j_2, j_1') \end{pmatrix},$$

$$\overline{\begin{pmatrix} k_1 & j_1 \\ j_2 & k_2 \end{pmatrix}} = \begin{pmatrix} k_2 & j_1 \\ j_2 & k_1 \end{pmatrix},$$

for all $k_1, k_2, k_1', k_2' \in k$ and $j_1, j_2, j_1', j_2' \in J$.

Skew-dimension one

- This is called a matrix structurable algebra, denoted by $M(J, 1)$.
- Similar construction for $M(J, \eta)$, $\eta \in k^\times$.
- Assume $\mathcal{A}$ skew-dimension one.
- Consider $0 \neq s_0 \in \mathcal{S}$, then $s_0^2 = \mu \cdot 1$, for some $\mu \in k^\times$.
- Allison-Faulkner '84: $\mathcal{A} \cong M(J, 1) \iff \mu$ is a square.
- In particular: There exists a quadratic field extension m/k such that $\mathcal{A} \otimes m \cong M(J, 1)$.
- Example cubic Jordan algebra: Any Albert algebra.
- Other example: Hermitian (3×3)-matrices over a composition algebra. These have dimension 6, 9, 15 or 27.

TKK Lie algebras

- Recall that $\mathrm{Instrl}(\mathcal{A}) := \langle V_{x,y} \mid x, y \in \mathcal{A} \rangle \leq \mathrm{End}(\mathcal{A})$ is a Lie subalgebra of $\mathfrak{g}l(\mathcal{A})$.
- If $\mathcal{A}$ is a structurable algebra, then

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$$

  is a 5-graded Lie algebra. Denote its i-th component by L_i.
- So $[L_i, L_j] \leq L_{i+j}$.
- $\mathcal{A}_+$ and $\mathcal{A}_-$ are two copies of $\mathcal{A}$.
- E.g.: $[x_+, y_-] = V_{x,y}$.
- E.g.: $[s_+, t_-] = L_s L_t$.
- E.g.: $[V_{x,y}, z_+] = (V_{x,y} z)_+$.

Exceptional Lie algebras

Definition.
A Lie algebra L is of (absolute) type X_n if $L \otimes \overline{k}$ is of type X_n.

Some examples:

Example.
We get the Freudenthal-Tits-magic square for the tensor product of two composition algebras:

$\dim(C_i)$	1	2	4	8
1	A_1	A_2	C_3	F_4
2	A_2	$A_2 \times A_2$	A_5	E_6
4	C_3	A_5	D_6	E_7
8	F_4	E_6	E_7	E_8

Exceptional Lie algebras

Example.

If J is a cubic Jordan algebra, we get (at least if $\mathrm{char}(k) = 0$):

$\dim(J)$	$\dim(M(J,1))$	type of $K(M(J,1))$
6	14	F_4
9	20	E_6
15	32	E_7
27	56	E_8

Definition.
An inner ideal of a Lie algebra L is a subspace I such that $[I,[I,L]] \leq I$. If I is 1-dimensional, any non-zero element of I is called extremal.

Any ideal is an inner ideal, 0 and L as well.

Example
For $L = sl_2$ we have a basis $\{e, f, h\}$ with $[e,f] = h$, $[h,e] = 2e$ and $[h,f] = -2f$. Then e is extremal by $[e,[e,h]] = 0 = [e,[e,e]]$ and $[e,[e,f]] = [e,h] = -2e$.

Canonical inner ideals

- $[L_i, [L_i, L_j]] \leq L_{2i+j}$.
- $\mathcal{S}_+$ will always be an inner ideal!
- If $\mathcal{S} = 0$, then $\mathcal{A}_+$ will always be an inner ideal.
- If $\mathcal{A}$ is central simple, any proper inner ideal $I \leq K(\mathcal{A})$ satisfies $[I, I] = 0$.
- A Lie algebra automorphism maps inner ideals on to inner ideals.

Some automorphisms

- For any $a \in \mathcal{A}$ and $s \in \mathcal{S}$ we get $\mathrm{ad}(a_+ + s_+)^5 = 0$.
- If $\mathcal{A}$ is central simple

$$e_+(a,s) := \exp(\mathrm{ad}(a_+ + s_+)) = \sum_{i=0}^{4} \frac{1}{i!} \mathrm{ad}(a_+ + s_+)^i$$

  is a Lie algebra automorphism. (Quite delicate in characteristic 5!)
- Set $E_+(\mathcal{A}) = \{e_+(a,s) \mid s \in \mathcal{S}, a \in \mathcal{A}\}$ and similarly $E_-(\mathcal{A})$.
- $E(\mathcal{A})$ is the subgroup of $\mathrm{Aut}(L)$ generated by $E_+(\mathcal{A})$ and $E_-(\mathcal{A})$.

Structurable division algebras

We define the operator U_x by $U_x(y) = V_{x,y}x$.

Definition.
An element $u \in \mathcal{A}$ is called conjugate invertible if there exists $\hat{u} \in \mathcal{A}$ such that $V_{u,\hat{u}} = \mathrm{Id}$. (This implies U_u invertible.)
The structurable algebra $\mathcal{A}$ is called division if every non-zero element of $\mathcal{A}$ is conjugate invertible.

Some examples:

- Associative division algebra with involution. Then $\hat{u} = \overline{u}^{-1}$.
- Albert division algebra.
- Jordan algebra associated with anisotropic quadratic form.

Some more exceptional division examples

- Octonion division algebra.
- Consider a quartic field extension whose splitting field has Galois group Alt(4) or Sym(4). Then one can apply a generalized Cayley-Dickson process to this algebra to obtain a skew-dimension one structurable division algebra (of dimension 8).
- Exceptional cases: quite hard to determine explicitly!

Inner ideals of $K(\mathcal{A})$

Theorem (De Medts - M.).
Let J be a central simple Jordan division algebra.
Then any non-trivial proper inner ideal of $K(J)$ distinct from J_- equals $e_-(j)(J_+)$ for a unique $j \in J$.

Note: $sl_2 = K(k)$.

Theorem (De Medts - M.).
Let $\mathcal{A}$ be a central simple structurable division algebra with $\mathcal{S} \neq 0$.
Then any non-trivial proper inner ideal of $K(\mathcal{A})$ distinct from $\mathcal{S}_-$ equals $e_-(a,s)(\mathcal{S}_+)$ for unique $a \in \mathcal{A}$ and $s \in \mathcal{S}$.

In particular: all inner ideals have the same dimension.

Associated geometry

Definition

Let X be a set and $\{U_x \mid x \in X\}$ a collection of subgroups of $\mathrm{Sym}(X)$. The data $(X, \{U_x\}_{x\in X})$ is a Moufang set if:

- For each $x \in X$, U_x fixes x and acts sharply transitively on $X \backslash \{x\}$.
- For each $g \in G := \langle U_x \mid x \in X\rangle \leq \mathrm{Sym}(X)$ and each $y \in X$ we have $g^{-1} U_y g = U_{y.g}$.

Corollary of previous result: If $\mathcal{A}$ is central simple division, the set of non-trivial proper inner ideals of $K(\mathcal{A})$ is a Moufang set.

Note: $U_{\mathcal{S}_-} = E_-(\mathcal{A})$.

Skew-dimension one structurable algebras

- Recall the skew-dimension one structurable algebra $M(J) := M(J, 1)$ with as elements
$$\begin{pmatrix} a & i \\ j & b \end{pmatrix},$$
with $a, b \in k$ and $i, j \in J$.
- Assumption: J is a cubic Jordan **division** algebra. ($N(j) = 0$ implies $j = 0$)
- Example of J: an Albert division algebra.
- Other example: k with cubic form $N(a) = a^3$.

Reduction

Lemma (De Medts - M.).
Any non-trivial inner ideal of $K(M(J))$ is the image of an inner ideal containing $\mathcal{S}_+$ under an element of $E(\mathcal{A})$.

- $\mathcal{S}_+$ is 1-dimensional.
- What are the inner ideals containing $\mathcal{S}_+$?
- The inner ideals of a structurable algebra $\mathcal{A}$ are the subspaces I satisfying $U_I(\mathcal{A}) \leq I$.
- In this case all non-trivial proper inner ideals of $M(J)$ are 1-dimensional and form a Moufang set.

A tale of two types of inner ideals

Lemma (De Medts - M.).
The only non-trivial inner ideals properly containing $\mathcal{S}_+$ are $\mathcal{S}_+ \oplus \langle a_+ \rangle$, with $\langle a \rangle$ an inner ideal.

- Each non-trivial proper inner ideal is 1 or 2-dimensional.
- If its dimension is 2, all subspaces are inner ideals.
- In particular: the Moufang set in $M(J)$ embeds in the lattice of inner ideals of $K(M(J))$.
- What geometry does this lattice of inner ideals form?
- First, we have to introduce some incidence-geometry concepts!

Point-line geometries

Definition.
A point-line geometry is a tuple $(\mathcal{P}, \mathcal{L})$, with $\mathcal{P}$ a set, called the set of points, and $\mathcal{L}$ a subset of $2^{\mathcal{P}}$, called the set of lines.

We say that a point $p \in \mathcal{P}$ is on a line $l \in \mathcal{L}$ if $p \in l$.

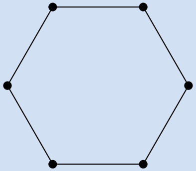
Definition.
We say that $(\mathcal{P}, \mathcal{L})$ is a partial linear space if for any two distinct $p_1, p_2 \in \mathcal{P}$ there exists at most one line containing both p_1 and p_2 and similarly, for any two distinct lines, there is at most one point on both lines.

Ordinary polygons

Example.

An ordinary n-gon.

Explicitly: $\mathcal{P} = \{1, \ldots, n\}$, $\mathcal{L} = \{\{1, 2\}, \{2, 3\}, \ldots, \{n, 1\}\}$.



Generalized polygons

Let $n \geq 3$. A generalized n-gon is a point-line geometry such that

- It is a partial linear space.
- It does not have ordinary m-gons as subgeometries, for all $2 < m < n$.
- It does have an ordinary n-gon as subgeometry.
- For any two points p_1, p_2, there exists an ordinary n-gon as subgeometry, containing both points.
- Similarly, for any two lines and any point and line, there exists an ordinary n-gon as subgeometry, containing both these elements.

We call a point-line geometry a generalized polygon if it is a generalized n-gon, for some $n \geq 3$.

Generalized triangles

Recall the (axiomatic) definition of a projective plane:

Definition.
A point-line geometry is a projective plane if

- Any two points are on a unique line.
- Any two lines intersect in a unique point.
- There are at least two lines.

Then one easily sees that a generalized 3-gon is precisely the same as a projective plane!

Moufang polygons

- Classifying projective planes is quite a tedious task! (Even if the point set is finite.)
- Classification is completed if one assumes thickness and some (transitivity) conditions on the automorphism group of the generalized polygon.
- This condition is called the Moufang condition, and the generalized polygons satisfying it the Moufang polygons.
- Implies $n = 3, 4, 6$ or 8.
- Classification due to Jacques Tits and Richard Weiss ('02).
- Now: describe some embeddings into Lie algebras, using inner ideals.

Moufang hexagon

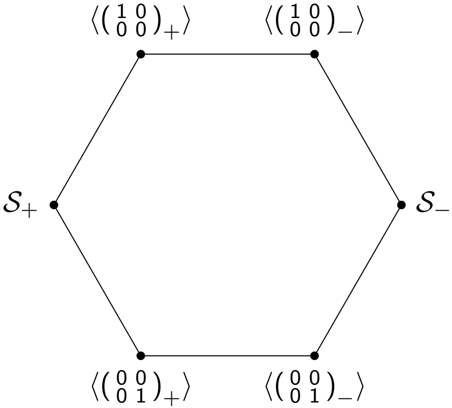
- A generalized hexagon is a point-line geometry which does not contain triangles, quadrangles or pentagons and such that any two distinct points, any two distinct lines and any point and line lie in a hexagon.
- In particular: two points are at distance at most 3.

Theorem (De Medts - M.).
The point-line geometry with as points the 1-dimensional inner ideals of $K(M(J))$, as lines all proper inner ideals of dimension > 1 and inclusion as incidence is a Moufang hexagon.

- $\mathcal{S}_+$ and $\mathcal{S}_-$ are at distance 3.

A hexagon

The generic hexagon in $K(M(J))$ is:



Some remarks

- In *Groups with Steinberg Relations and Coordinatization of Polygonal Geometries* ('77) Faulkner constructs a Lie algebra starting from a cubic Jordan division algebra. He then constructs a generalized hexagon with as points and lines some specific inner ideals.
- Similar ideas can also be used to determine inner ideals of some other TKK Lie algebras that will correspond to projective planes. We will describe this now.

Alternative algebras

- Let F be an alternative algebra (i.e., satisfying $x^2y = x(xy)$ and $yx^2 = (yx)x$).
- Consider $\mathcal{A} := F \oplus F$, with multiplication $(x, y)(a, b) = (xa, by)$ and involution $(x, y) \mapsto (y, x)$.
- Then $\mathcal{A}$ is a structurable algebra.
- $\mathcal{S} = \{(f, -f) \mid f \in F\}$ has the same dimension as F.
- So, in general not a skew-dimension one structurable algebra.
- Assume from now on that F is division.

Again a reduction

Lemma (De Medts - M.).
Any non-trivial inner ideal of $K(\mathcal{A})$ is the image of an inner ideal containing $\mathcal{S}_+$ under an element of $E(\mathcal{A})$.

- What are the inner ideals containing $\mathcal{S}_+$?
- There are precisely two of these: $\mathcal{S}_+ \oplus (F \oplus 0)_+$ and $\mathcal{S}_+ \oplus (0 \oplus F)_+$.
- Hence all non-trivial proper inner ideals have dimension $\dim(F)$ or $2\dim(F)$.

Constructing a geometry

- $\mathcal{P} = \{I \mid I \text{ is a minimal non-trivial inner ideal}\}$
- $\mathcal{L} = \{I \mid I \text{ is a proper non-trivial non-minimal inner ideal}\}$.
- Then $(\mathcal{P}, \mathcal{L})$ is a point-line geometry with containment as incidence.
- The situation is more involved, since the minimal inner ideals are not 1-dimensional.

Lemma (De Medts-M.).
Let $I, J \in \mathcal{P}$ be two distinct points at distance d in $(\mathcal{P}, \mathcal{L})$. Then

$$
\begin{aligned}
d = 1 &\iff [I, J] = 0, \\
d = 2 &\iff [I, J] \neq 0 \text{ and } [I, [I, J]] = 0, \\
d = 3 &\iff [I, [I, J]] \neq 0.
\end{aligned}
$$

Thin generalized hexagon

We call a geometry thin, if every point is on precisely two lines.

Theorem (De Medts-M.).
The point-line geometry $(\mathcal{P}, \mathcal{L})$ is a thin generalized hexagon. It is the so-called double of a Moufang triangle.

Moufang quadrangles?

- For some classical class of Moufang quadrangles we also have a description in terms of inner ideals. (Related to quadratic forms.)
- There are also Moufang quadrangles of type E_6, E_7 and E_8.
- It seems that one obtains this as inner ideal geometry if you consider a suitable structurable algebra.
- Hopefully, we can attack higher rank geometries with this approach as well.
- This will be related to extremal geometries.

Extremal geometries

From now on: joint with Hans Cuypers (TU Eindhoven)

- Recall: $0 \neq x \in L$ is extremal if $[x,[x,L]] \leq \langle x \rangle$.
- E = set of extremal elements.
- x is a zero divisor if $[x,[x,L]] = 0$.
- If L does not contain a non-zero zero divisor, it is called non-degenerate.
- Assume that L is a simple non-degenerate Lie algebra generated by its extremal elements.
- $\mathcal{P} = \{\langle x \rangle \mid x \in E\}$.
- $\mathcal{L} = \{\langle x, y \rangle \mid [x,y] = 0$ and $\lambda x + \mu y \in E, \forall \lambda, \mu \in k, (\lambda, \mu) \neq (0,0)\}$.
- $\Gamma := (\mathcal{P}, \mathcal{L})$ is a point-line geometry, called the extremal geometry.
- Introduced by Arjeh Cohen and Gabor Ivanyos. ('06)

Results Cohen and Ivanyos

Theorem.
If $\mathcal{L} \neq \emptyset$, then Γ is isomorphic to a root shadow space of type $\mathrm{A}_{n,\{1,n\}}$ ($n \geq 2$), $\mathrm{BC}_{n,2}$ ($n \geq 3$), $\mathrm{D}_{n,2}$ ($n \geq 4$), $\mathrm{E}_{6,2}$, $\mathrm{E}_{7,1}$, $\mathrm{E}_{8,8}$, $\mathrm{F}_{4,1}$ or $\mathrm{G}_{2,2}$.

- Precise definition does not matter. What is important: they are classified and more or less known.
- Example: $A_{n,\{1,n\}}$ is associated with a projective space.
- Example: $G_{2,2}$ are precisely these generalized hexagons.
- The extremal geometry in a split Lie algebra of type X_n has the same type.
- Moreover, their approach works in any characteristic.
- However, there is one root shadow space missing, namely the polar spaces (generalized quadrangles are of this type).

Inner line ideal geometry

- In order to fix this, we extended the line set.
- We call I an inner line ideal if it is a proper inner ideal containing two linearly independent extremal elements and is minimal with these properties.
- Set $\mathcal{L}' := \{I \mid I \text{ is an inner line ideal}\}$.
- We call $\Gamma' = (\mathcal{P}, \mathcal{L}')$ the inner line ideal geometry of L.

Theorem (Cuypers-M.).
Suppose L is a simple non-degenerate Lie algebra generated by extremal elements over a field of characteristic not 2. Then we have one of the following:

(a) The extremal geometry of L contains lines and coincides with the inner line ideal geometry.

(b) The extremal geometry of L contains no lines, but L contains two commuting, but linearly independent extremal elements; the inner line ideal geometry is a non-degenerate polar space of rank at least 2.

(c) The Lie algebra L does not contain commuting, but linearly independent extremal elements, and the inner line ideal geometry has no lines.

Connection with structurable algebras?

Relying on results of Stavrova:

Theorem (Cuypers - M.).
If L is a non-degenerate non-symplectic simple Lie algebra generated by its extremal elements, then $L = K(\mathcal{A})$ for some skew-dimension one structurable algebra $\mathcal{A}$.

- Moreover, the inner line ideal geometry coincides with the extremal geometry if and only if $\mathcal{A}$ is isomorphic to a matrix structurable algebra.
- If the inner line ideal geometry has no lines, it is coming from a skew-dimension one structurable division algebra. Hence, it is a Moufang set!

Some remarks

- In particular one sees that one does not recover all Moufang sets, using the extremal geometry/inner line ideal geometry approach.
- A geometry of type $A_{2,\{1,2\}}$ is precisely the double of a projective plane.
- So we only get the double of the projective plane over the ground field as extremal geometry.
- For other projective planes, the minimal inner ideals need to have dimension strictly bigger than 1.

Some more questions

- What happens in characteristic 2 (and 3)? Extremal elements in characteristic 2 are well-defined, what is "good" definition for inner ideals of Lie algebras?
- Inner ideals of a Jordan algebra correspond to inner ideals of its Lie algebra, what happens when structurable algebra is not Jordan?
- Good definition for structurable algebras in characteristic 2 and/or 3?
- Explicit construction for forms of matrix structurable algebras corresponding to quadrangles of type E_6, E_7, E_8?

Thanks for your attention!