# Structurable algebras, TKK Lie algebras and their inner ideals

Jeroen Meulewaeter
30 april 2019

UNIVERSITEIT GENT

Joint work with Tom De Medts

*II International Workshop on Non-Associative Algebras in Porto*

## 2 Inner ideals

**Assumption**

All algebras are finite-dimensional and defined over a field $k$ of characteristic different from 2 and 3!

**Definition**

An inner ideal of a Lie algebra $L$ is a subspace $I$ such that $[I,[I,L]] \leq I$. If $I$ is 1-dimensional, any non-zero element of $I$ is called extremal.

**Example**

For $L = sl_2$ we have a basis $\{e, f, h\}$ with $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. Then $e$ is extremal by $[e,[e,h]] = 0 = [e,[e,e]]$ and $[e,[e,f]] = [e,h] = -2e$.

# 3 TKK Lie algebras

- If $\mathcal{A}$ is a structurable algebra, then

$$K(\mathcal{A}) = \mathcal{S}_- \oplus \mathcal{A}_- \oplus \mathrm{Instrl}(\mathcal{A}) \oplus \mathcal{A}_+ \oplus \mathcal{S}_+$$

  is a 5-graded Lie algebra. Denote its $i$-th component by $L_i$.
- E.g.: $[x_+, y_-] = V_{x,y}$.
- $[L_i, [L_i, L_j]] \leq L_{2i+j}$.
- $\mathcal{S}_+$ will always be an inner ideal!
- If $\mathcal{S} = 0$, then $\mathcal{A}_+$ will always be an inner ideal.
- If $\mathcal{A}$ is central simple, any non-trivial inner ideal $I \leq K(\mathcal{A})$ satisfies $[I, I] = 0$.

# 4 Some automorphisms

- A Lie algebra automorphism maps inner ideals on to inner ideals.
- For any $a \in \mathcal{A}$ and $s \in \mathcal{S}$ we get $\mathrm{ad}(a_+ + s_+)^5 = 0$.
- If $\mathcal{A}$ is central simple

$$e_+(a,s) := \exp(\mathrm{ad}(a_+ + s_+)) = \sum_{i=0}^{4} \frac{1}{i!} \mathrm{ad}(a_+ + s_+)^i$$

  is a Lie algebra automorphism.
- Set $E_+(\mathcal{A}) = \{e_+(a,s) \mid s \in \mathcal{S}, a \in \mathcal{A}\}$ and similarly $E_-(\mathcal{A})$.
- $E(\mathcal{A})$ is the subgroup of $\mathrm{Aut}(L)$ generated by $E_+(\mathcal{A})$ and $E_-(\mathcal{A})$.

# 5 Inner ideals of Jordan algebras

**Definition**

A subspace $I$ of a Jordan algebra $J$ is an inner ideal if $U_I(J) \leq I$.

**Example**

Let $J = \mathrm{Jord}(Q, c)$ be the Jordan algebra corresponding to a non-degenerate quadratic form $Q$ with basepoint $c$.
The non-trivial inner ideals are precisely the isotropic subspaces.

# 6 Some remarks

- If $A$ is an associative algebra, $A^+ = (A, \circ)$ is a Jordan algebra with $a \circ b = (ab + ba)/2$.
- Motivation for terminology: $I \leq A^+$ is inner if $IAI \leq I$.
- Studied by McCrimmon in 1971 (*Inner ideals in quadratic Jordan algebras*).
- If $J$ is division, there are no non-trivial inner ideals.
- A subspace $I \leq J$ is an inner ideal if and only if $I_+$ is an inner ideal in $K(J)$.

# 7 Structurable division algebras

**Theorem (De Medts - M.)**

*Let $J$ be a central simple Jordan division algebra.*
*Then any non-trivial inner ideal of $K(J)$ distinct from $J_-$ equals $e_-(j)(J_+)$ for a unique $j \in J$.*

Note: $\mathfrak{sl}_2 = K(k)$.

**Theorem (De Medts - M.)**

*Let $\mathcal{A}$ be a central simple structurable division algebra with $\mathcal{S} \neq 0$.*
*Then any non-trivial inner ideal of $K(\mathcal{A})$ distinct from $\mathcal{S}_-$ equals $e_-(a, s)(\mathcal{S}_+)$ for unique $a \in \mathcal{A}$ and $s \in \mathcal{S}$.*

# 8 Moufang sets

**Definition**

Let $X$ be a set and $\{U_x \mid x \in X\}$ a collection of subgroups of $\mathrm{Sym}(X)$. The data $(X, \{U_x\}_{x \in X})$ is a Moufang set if:

- For each $x \in X$, $U_x$ fixes $x$ and acts sharply transitively on $X \backslash \{x\}$.
- For each $g \in G := \langle U_x \mid x \in X \rangle$ and each $y \in X$ we have $U_y^g = U_{y.g}$.

If $\mathcal{A}$ is central simple division, the set of non-trivial inner ideals of $K(\mathcal{A})$ is a Moufang set.
Note: $U_{\mathcal{S}_-} = E_-(\mathcal{A})$.

# 9 Skew-dimension one structurable algebras

- Recall the skew-dimension one structurable algebra with as elements
$$\begin{pmatrix} a & i \\ j & b \end{pmatrix},$$
with $a, b \in k$ and $i, j \in J$.
- We denote it by $M(J)$.
- $J$ is a cubic Jordan **division** algebra.
- Example of $J$: an Albert division algebra.
- Other example: $k$ with cubic form $N(a) = a^3$.

# Reduction

**Lemma (De Medts - M.)**

*Any non-trivial inner ideal of $K(M(J))$ is the image of an inner ideal containing $\mathcal{S}_+$ under an element of $E(\mathcal{A})$.*

- $\mathcal{S}_+$ is 1-dimensional.
- What are the inner ideals containing $\mathcal{S}_+$?
- The inner ideals of a structurable algebra $\mathcal{A}$ are the subspaces $I$ satisfying $U_I(\mathcal{A}) \leq I$.
- In this case all non-trivial inner ideals of $M(J)$ are 1-dimensional and form a Moufang set.

## 11 A tale of two types of inner ideals

**Lemma (De Medts - M.)**

*The only non-trivial inner ideals properly containing $\mathcal{S}_+$ are $\mathcal{S}_+ \oplus \langle a_+ \rangle$, with $\langle a \rangle$ an inner ideal.*

- Each non-trivial inner ideal is 1 or 2-dimensional.
- If its dimension is 2, all subspaces are inner ideals.
- In particular: the Moufang set in $M(J)$ embeds in the lattice of inner ideals of $K(M(J))$.

# 12 Generalized hexagon

- A generalized hexagon is a point-line geometry which does not contain triangles, quadrangles or pentagons and such that any two distinct points, any two distinct lines and any point and line lie in a hexagon.
- In particular: two points are at distance at most 3.

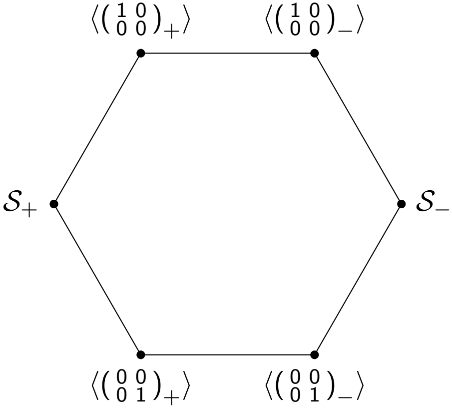
**Theorem (De Medts - M.)**

*The point-line geometry with as points the 1-dimensional inner ideals, as lines all non-trivial inner ideals of dimension $> 1$ and inclusion as incidence is a generalized hexagon.*

- $\mathcal{S}_+$ and $\mathcal{S}_-$ are at distance 3.

# 13 A hexagon

The generic hexagon in $K(M(J))$ is:



# 14 Some remarks

- In *Groups with Steinberg Relations and Coordinatization of Polygonal Geometries* ('77) Faulkner constructs a Lie algebra starting from a cubic Jordan division algebra. He then constructs a generalized hexagon with as points and lines some specific inner ideals.
- Similar ideas can also be used to determine inner ideals of some other structurable algebras and TKK Lie algebras that will correspond to projective planes.

# 15 Extremal geometries

Joint with Hans Cuypers (TU Eindhoven)

- Relying on results of Stavrova:

**Theorem (Cuypers - M.)**

*If $L$ is a non-degenerate non-symplectic simple Lie algebra generated by its extremal elements, then $L = K(\mathcal{A})$ for some skew-dimension one structurable algebra $\mathcal{A}$.*

- Cohen and Ivanyos: construction of point-line geometries called extremal geometries with as points extremal elements.
- Currently: trying to see the lines of these geometries as minimal inner ideals containing extremal elements and extend the results of Cohen and Ivanyos.

# 16 Open questions

- What happens in characteristic 2 (and 3)? Extremal elements in characteristic 2 are well-defined, what is "good" definition for inner ideals of Lie algebras?
- Inner ideals of a Jordan algebra correspond to inner ideals of its Lie algebra, what happens when structurable algebra is not Jordan?
- Which class of structurable algebras corresponds to rank 2 geometries?

Thanks for your attention!