EOS RESEARCH PROJECT: synopsis

"High-dimensional expanders and Kac–Moody–Steinberg groups"

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State of the art

The study of *expander graphs* originates from computer science and can be traced back to Pinsker in 1973 [Pin73]. An expander graph is, roughly speaking, a sparse but nevertheless sufficiently connected graph, making it useful for constructing computer networks that are reliable as well as cost-effective. A huge area of research has emerged from this, involving both mathematics and computer science (see [Lub12] and references therein). Two excellent surveys are [HLW06, Lub12].

More recently, the notion of expander graphs has been extended to higher dimensions, giving rise to (families of) simplicial complexes called *high-dimensional expanders*. Their study provides the main motivation for this research proposal. We refer to the excellent survey [Lub18].

A recurring theme of many of the known constructions of high-dimensional expanders is the use of *Bruhat–Tits buildings* of algebraic groups over local fields, the expanders being obtained as suitable quotients of such buildings by cocompact arithmetic lattices.

Our ambition with this project is to contribute to the state-of-the-art by providing new abundant sources of examples arising from simplicial complexes associated with a non-classical family of groups, called *Kac–Moody–Steinberg groups* (*KMS groups* for short).

Scientific research hypothesis and objectives

KMS groups provide a remarkable but little studied class of finitely presented groups which is interesting in its own right. They are defined as fundamental groups of certain complexes of finite p-groups, where p is a fixed prime number. The finite p-groups in question are the positive unipotent subgroups of basic Levi subgroups of spherical type in a 2-spherical *Kac–Moody group* over a finite field. One example of a KMS group is the group $\mathcal{U}$ with presentation

$$\mathcal{U} = \langle x_1, \dots, x_d \mid x_i^p,\ [[x_i, x_j], x_j]\ (i \neq j)\rangle.$$

In this project, we make the hypothesis that, by unveiling the combinatorial, geometric, Lie-algebraic and cohomological properties of KMS groups, we will be able to provide key examples relevant not only to the vibrant emerging theory of high-dimensional expanders, but also to various other core problems in geometric group theory concerning the residual properties of hyperbolic groups, lattice envelopes, cohomology vanishing and group stability. The starting point of this novel approach is inspired by the recent preliminary work in [KO23] and [CCKW22, Sec. 7]. The fact that KMS groups provide a source of HDX's is substantiated in [dPVB24].

Regarding the theory of high-dimensional expanders, the main contribution of the project will be achieved by the following milestone.

> **Milestone.** *Use KMS groups to construct families of high-dimensional expanders of* **bounded degree** *associated with finite simple Lie type groups of* **unbounded rank** *(resp. with alternating groups of* **unbounded degree***).*

The milestone will be obtained as a byproduct of our systematic investigations of KMS groups, which are articulated around various precise research questions organized into 3 Work Packages: (WP1) Algebraic structure; (WP2) Geometric structure; (WP3) Lie-algebraic methods. The cross-sectional objectives of the proposal across these WPs can be summarized as follows:

- Study the global structure of KMS groups and the associated geometries.
- Study the local structure of KMS groups and the associated geometries.
- Study the algebraic quotients of KMS groups.
- Study the spectral and cohomological properties of KMS groups.

This project addresses cutting-edge research topics with a lot to be discovered, but at the same time these topics have already shown their potential. We refer to [dPVB24] and [CM24] for intermediate results already established within the project framework. We believe that our investigations will have a large impact not only on the specific research questions that will be investigated in order to reach the Milestone, but that our contributions in this context will moreover be the starting point for future research projects in geometric group theory.

Research team

The project consortium involves 3 PhD students and several post-docs. We expect a close interaction between all members involved in this project. The project coordination relies on regular workshops and "sync events" gathering all members of the consortium, as well as two larger conferences. The first of those was a Summer School that took place in Ghent in May 2023 (see `https://algebra.ugent.be/hdx/`). The second one will be a research conference that will be held towards the end of the project.

References

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