Suzuki-Ree groups as algebraic groups over $\mathbb{F}_{\sqrt{p}}$

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Overview

1 — Finite simple groups

Finite simple groups

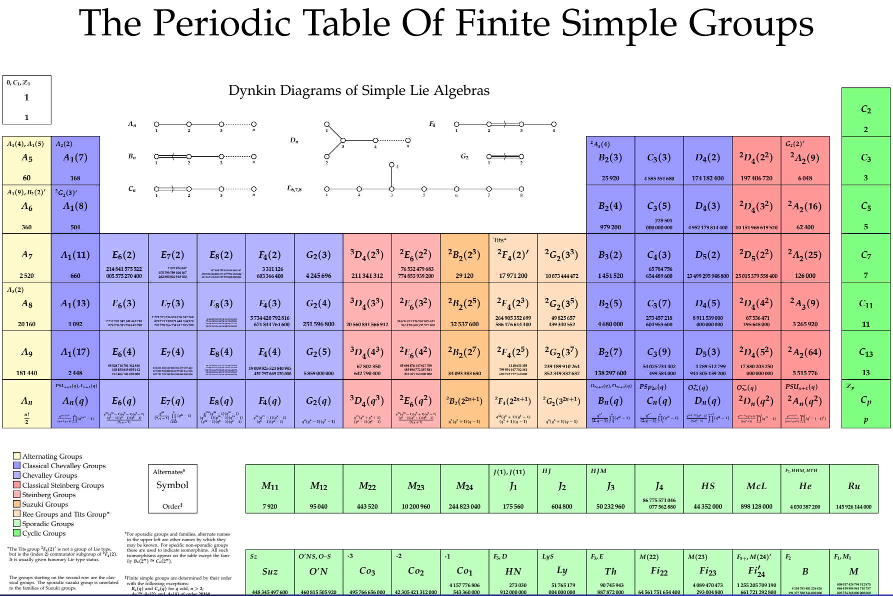
Theorem

Every finite simple group belongs to at least one of these classes:

- *Cyclic groups* $\leftarrow$ *abelian*
- *Sporadic groups* $\leftarrow$ *finite in number*
- *Alternating groups* $\leftarrow$ *well understood*
- *Groups of Lie type* $\leftarrow$ *many subclasses*

Proof.

Omitted. $\square$


The Periodic Table Of Finite Simple Groups
Dynkin Diagrams of Simple Lie Algebras
A_n B_n C_n D_n $E_{6,7,8}$ F_4 G_2
0, C_1, $\mathbb{Z}_1$ 1 1
A_5 60
$A_1(7)$ 168
A_6 360
$A_1(8)$ 504
A_7 2 520
$A_1(11)$ 660
$E_6(2)$
$E_7(2)$
$E_8(2)$
$F_4(2)$ 3 311 126 603 366 400
$G_2(3)$ 4 245 696
${}^3D_4(2^3)$ 211 341 312
${}^2E_6(2^2)$ 76 532 479 683 774 853 939 200
${}^2B_2(2^3)$ 29 120
${}^2F_4(2)'$ 17 971 200
${}^2G_2(3^3)$ 10 073 444 472
$B_2(3)$ 25 920
$C_3(3)$ 4 585 351 680
$D_4(2)$ 174 182 400
${}^2D_4(2^2)$ 197 406 720
${}^2A_2(9)$ 6 048
$B_2(4)$ 979 200
$C_3(5)$ 228 501 000 000 000
$D_4(3)$ 4 952 179 814 400
${}^2D_4(3^2)$ 10 151 968 619 520
${}^2A_2(16)$ 62 400
$B_3(2)$ 1 451 520
$C_4(3)$ 65 784 756 654 489 600
$D_5(2)$ 23 499 295 948 800
${}^2D_5(2^2)$ 25 015 379 558 400
${}^2A_2(25)$ 126 000
A_8 20 160
$A_1(13)$ 1 092
$E_6(3)$
$E_7(3)$
$E_8(3)$
$F_4(3)$ 5 734 420 792 816 671 844 761 600
$G_2(4)$ 251 596 800
${}^3D_4(3^3)$ 20 560 831 566 912
${}^2E_6(3^2)$
${}^2B_2(2^5)$ 32 537 600
${}^2F_4(2^3)$ 264 905 352 699 586 176 614 400
${}^2G_2(3^5)$ 49 825 657 439 340 552
$B_2(5)$ 4 680 000
$C_3(7)$ 273 457 218 604 953 600
$D_4(5)$ 8 911 539 000 000 000 000
${}^2D_4(4^2)$ 67 536 471 195 648 000
${}^2A_3(9)$ 3 265 920
A_9 181 440
$A_1(17)$ 2 448
$E_6(4)$
$E_7(4)$
$E_8(4)$
$F_4(4)$ 19 009 825 523 840 945 451 297 669 120 000
$G_2(5)$ 5 859 000 000
${}^3D_4(4^3)$ 67 802 350 642 790 400
${}^2E_6(4^2)$
${}^2B_2(2^7)$ 34 093 383 680
${}^2F_4(2^5)$
${}^2G_2(3^7)$ 239 189 910 264 352 349 332 632
$B_2(7)$ 138 297 600
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${}^2D_4(5^2)$ 17 880 203 250 000 000 000
${}^2A_2(64)$ 5 515 776
A_n $\frac{n!}{2}$
$A_n(q)$ $E_6(q)$ $E_7(q)$ $E_8(q)$ $F_4(q)$ $G_2(q)$ ${}^3D_4(q^3)$ ${}^2E_6(q^2)$ ${}^2B_2(2^{2n+1})$ ${}^2F_4(2^{2n+1})$ ${}^2G_2(3^{2n+1})$ $B_n(q)$ $C_n(q)$ $D_n(q)$ ${}^2D_n(q^2)$ ${}^2A_n(q^2)$
C_2 2
C_3 3
C_5 5
C_7 7
C_{11} 11
C_{13} 13
$\mathbb{Z}_p$ C_p p
Alternating Groups
Classical Chevalley Groups
Chevalley Groups
Classical Steinberg Groups
Steinberg Groups
Suzuki Groups
Ree Groups and Tits Group*
Sporadic Groups
Cyclic Groups
Alternates† Symbol Order‡
M_{11} 7 920
M_{12} 95 040
M_{22} 443 520
M_{23} 10 200 960
M_{24} 244 823 040
J_1 175 560
J_2 604 800
J_3 50 232 960
J_4 86 775 571 046 077 562 880
HS 44 352 000
McL 898 128 000
He 4 030 387 200
Ru 145 926 144 000
Suz 448 345 497 600
$O'N$ 460 815 505 920
Co_3 495 766 656 000
Co_2 42 305 421 312 000
Co_1 4 157 776 806 543 360 000
HN 273 030 912 000 000
Ly 51 765 179 004 000 000
Th 90 745 943 887 872 000
Fi_{22} 64 561 751 654 400
Fi_{23} 4 089 470 473 293 004 800
Fi'_{24} 1 255 205 709 190 661 721 292 800
B
M
*The Tits group ${}^2F_4(2)'$ is not a group of Lie type, but is the (index 2) commutator subgroup of ${}^2F_4(2)$. It is usually given honorary Lie type status.
The groups starting on the second row are the classical groups. The sporadic suzuki group is unrelated to the families of Suzuki groups.
†For sporadic groups and families, alternate names in the upper left are other names by which they may be known. For specific non-sporadic groups these are used to indicate isomorphisms. All such isomorphisms appear on the table except the family $B_n(2^m) \cong C_n(2^m)$.
‡Finite simple groups are determined by their order with the following exceptions: $B_n(q)$ and $C_n(q)$ for q odd, $n > 2$;

2 — Groups of Lie type

Groups of Lie type: a big question

Who is responsible for the structure within the groups of Lie type?

Groups of Lie type: origin in algebraic group theory

Definition (Roughly)

An algebraic group is a machine that manufactures groups (from rings).

$$G : (\mathbf{ring}) \to (\mathbf{group}) : R \rightsquigarrow G(R)$$

- More generally: an algebraic k-group manufactures groups from k-algebra's.

$$G : (\mathbf{k\text{-}alg}) \to (\mathbf{group}) : K \rightsquigarrow G(K)$$

Groups of Lie type: algebraic groups

Idea

It is better to study the machine itself than what it produces.

- Ideally: few algebraic groups produce all Lie type FSGs.

$$G : (\textbf{fin-field}) \to (\textbf{fsg}) : \mathbb{F}_q \rightsquigarrow G(q)$$

Groups of Lie type: the Chevalley groups

Consider PSL:

- There is a class of finite simple groups $\mathsf{PSL}_n(q)$.
- There is no class of algebraic groups PSL_n!
- However there are algebraic groups SL_n, PGL_n and a surjection $\mathsf{SL}_n \to \mathsf{PGL}_n$ and then

$$\mathsf{PSL}_n(q) = \mathrm{im}(\mathsf{SL}_n(q) \to \mathsf{PGL}_n(q))$$

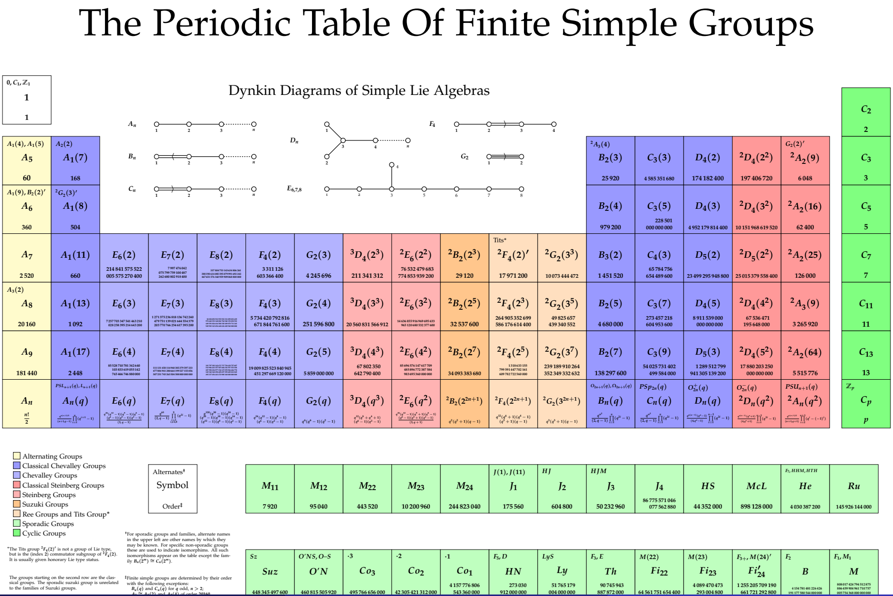
- So in better approximation:

finite simple groups of Lie type

$\Longleftrightarrow$

isogeny classes of semi-simple algebraic groups

- This deals with all the ordinary classes.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An n!/2
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C7 7
C11 11
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Groups of Lie type: the Steinberg groups

- It is still the case that every such group arises (roughly) as $G(q)$ (for some G and q).
- But not uniformely! For instance there is an algebraic $\mathbb{F}_p$-group G such that

$$G(p) = \mathrm{PSU}(p^2/p) \text{ but then}$$
$$G(p^2) = \mathrm{PSL}(p^2) \quad !!!$$

- ... and we need a different algebraic $\mathbb{F}_{p^2}$-group G' such that

$$G'(p^2) = \mathrm{PSU}(p^4/p^2) \text{ but then}$$
$$G'(p^4) = \mathrm{PSL}(p^4) \quad !!!$$

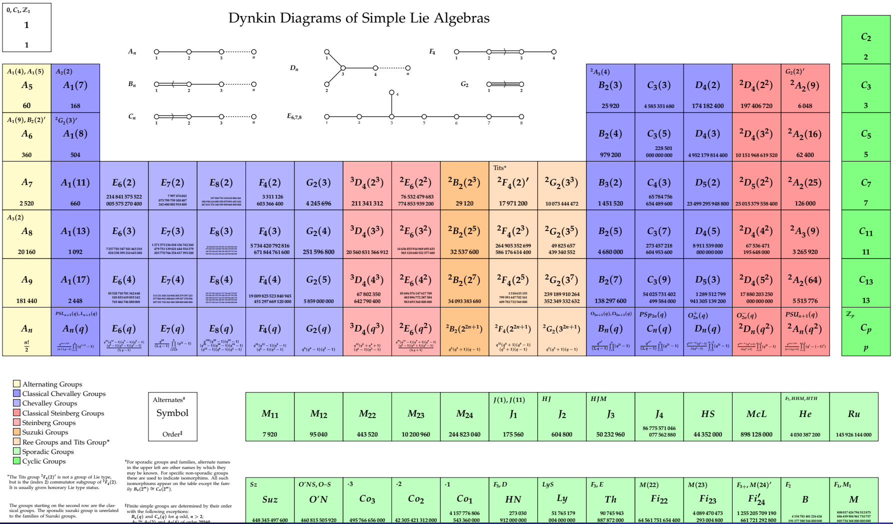
- This is quite annoying but well understood.

Groups of Lie type: the Steinberg groups

- This is quite annoying but well understood.
- We understand PSU as a form of PGL.
- On sidenote, it is weird that not all PSU's form a single family.
- Fake class?

Groups of Lie type

The Periodic Table Of Finite Simple Groups



3 — Suzuki-Ree groups

Suzuki-Ree groups: brief history

- Suzuki was classifying a class of finite groups.
- Found strange guys $G(2^{2e+1})$: the Suzuki groups.
- Ree: they are 'twists' of the Lie-type group B_2, notation ${}^2\mathrm{B}_2(2^{2e+1})$.
- Ree also found ${}^2\mathrm{G}_2(3^{2e+1})$ and ${}^2F_4(2^{2e+1})$; the small and large Ree groups.

A disturbance in the force

- Only defined over 'very few' fields!
- There is *no* algebraic group such that $G(2^{e'}) = {}^2\mathrm{B}_2(2^{2e+1})$ in a natural manner

families of Suzuki groups ${}^2B_2(2^{2n+1})$ and Ree groups ${}^2F_4(2^{2n+1})$. Similarly, the Dynkin diagram G_2 in perfect fields of characteristic three has an automorphism that swaps the short and long root, and if $q = 3\tilde{q}^2$ leads to the final class of Ree groups, ${}^3G_2(3^{2n+1})$. In contrast to the Steinberg groups, the Suzuki-Ree groups cannot be easily viewed as algebraic groups over a suitable subfield; morally, one "wants" to view ${}^2B_2(2^{2n+1})$ and ${}^2F_4(2^{2n+1})$ as being algebraic over the field of $2^{n+1/2}$ elements (and similarly view ${}^3G_2(3^{2n+1})$ as algebraic over the field of $3^{n+1/2}$ elements), but such fields of course do not exist. (Despite superficial similarity, this issue appears unrelated to the "field with one element" discussed in Remark 3, although both phenomena do suggest that there is perhaps a useful generalisation of the concept of a field that is currently missing from modern mathematics.) One can also view the Steinberg and Suzuki-Ree

Blog Terence Tao: algebraic groups over $\mathbb{F}_{\sqrt{2^{2n+1}}}$?

Suzuki-Ree groups: $\mathbb{F}_{\sqrt{p}}$

20. SUZUKI AND REE GROUPS

Lie type	Order of group	q^2
${}^2\mathrm{B}_2(q^2)$	$q^4(q^2-1)(q^4+1)$	2^{2r+1}
${}^2\mathrm{G}_2(q^2)$	$q^6(q^2-1)(q^6+1)$	3^{2r+1}
${}^2\mathrm{F}_4(q^2)$	$q^{24}(q^2-1)(q^6+1)(q^8-1)(q^{12}+1)$	2^{2r+1}

TABLE 1. Suzuki and Ree groups

Book Jim Humphreys: $q^2 = 2^{2r+1}$?

Suzuki-Ree groups: construction

Ingredients:

- A perfect ground field k of characteristic $p = 2, 3, 2$
- A square root of the Frobenius $\sigma : k \to k$, i.e. $\sigma \circ \sigma = \mathrm{fr}_k$:

$$\sigma(\sigma(x)) = x^p = \mathrm{fr}(x).$$

  (Thus if k is finite, it must be $\mathbb{F}_2^{2e+1}$ or $\mathbb{F}_3^{2e+1}$.)
- A semi-simple $\mathbb{F}_p$-group G of type B_2, G_2, F_4.
- An isogeny $\pi : G \to G$ such that $\pi \circ \pi = \mathrm{Fr}_G$ (the Frobenius on G)

Suzuki-Ree groups: construction

Ingredients:

- A square root of the Frobenius $\sigma : k \to k$, i.e. $\sigma \circ \sigma = \mathrm{fr}_k$
- An isogeny $\pi : G \to G$ such that $\pi \circ \pi = \mathrm{Fr}_G$ (the Frobenius on G)
- There is an involution $\alpha : G(k) \to G(k) : u \mapsto \pi \circ u \circ \sigma^{-1}$.

$$\begin{array}{ccc} \operatorname{Spec} k & \overset{\alpha(u)}{\dashrightarrow} & G \\ {\scriptstyle \sigma^{-1}} \downarrow & & \uparrow {\scriptstyle \pi} \\ \operatorname{Spec} k & \overset{u}{\longrightarrow} & G \end{array} \quad \text{or} \quad \begin{array}{ccc} k & \underset{\alpha(u)}{\dashleftarrow} & k[G] \\ {\scriptstyle \sigma^{-1}} \uparrow & & \downarrow {\scriptstyle \pi} \\ k & \underset{u}{\longleftarrow} & k[G] \end{array}$$

- Define: ${}^2G(k, \sigma) = \mathrm{Fix}(\alpha) \leq G(k)$.
- (Also defined over imperfect fields, by Tits.)

4 — More disturbances

More disturbances: mixed groups

- Tits invented buildings to study and classify algebraic groups.
- Tits classified spherical buildings of rank ≥ 3 buildings. The 'ideal correspondence' is:

$$\text{buildings} \iff \text{semi-simple algebraic groups}$$

- Found strange guys $G(k, \ell)$: the mixed groups.
- They seem to be algebraic groups 'defined over two fields'!

More disturbances: construction of mixed groups

Ingredients:

- Field extensions $k^p \subseteq \ell \subseteq k$;
- A k-group G (semi-simple, adjoint) of type $\mathrm{B}_n, \mathrm{C}_n, \mathrm{G}_2, \mathrm{F}_4$ with root system Φ.
- Tits' theory provides a collection of generators for $G(k)$ namely (ignoring the torus)

$$G(k) = \langle u_r(t) \mid r \in \Phi, t \in k \rangle$$

  the group $G(k)$ is described by commutation relations which look like $[u_r(x), u_s(y)] = u_r(x)u_{r+2s}(x^2y)$
- Tits observed that there is an interesting subgroup

$$G(\ell, k) = \left\langle u_r(t) \mid r \in \Phi, \begin{cases} t \in k & r \text{ long} \\ t \in \ell & r \text{ short} \end{cases} \right\rangle$$

More disturbances: mixed groups as indifferent groups

Tits calls them indifferent because

$$\cdots \subseteq k^p \subseteq \ell^p \subseteq k \subseteq \ell \subseteq k^{1/p} \subseteq \cdots$$

$$\mathrm{B}_n(k,\ell) \cong \mathrm{C}_n(\ell^2, k)$$

$$\mathrm{G}_2(k,\ell) \cong \mathrm{G}_2(\ell^3, k)$$

$$\mathrm{F}_4(k,\ell) \cong \mathrm{F}_4(\ell^2, k)$$

Both fields should play the same role.
Furthermore we have embeddings

$$\mathrm{X}(k,\ell) \subseteq \mathrm{X}(\ell,\ell) = \mathrm{X}(\ell)$$
$$\mathrm{X}(k,\ell) \subseteq \mathrm{X}(k,k^{1/p}) \cong \mathrm{X}'(k,k) = \mathrm{X}'(k)$$

So $\mathrm{X}(k,\ell)$ is a *mixture* of $X(\ell)$ and $\mathrm{X}'(k)$ somehow.

More disturbances: Weiss' quadrangles

- Weiss was classifying Moufang polygons (another class of buildings)
- Found $\mathrm{WQ}_4(k, \ell)$: very strange groups.
- Defined over $k^2 \subsetneq \ell \subsetneq k$: imperfect fields only!
- The construction is ... technical.
- Intuitively, they should be forms of $\mathrm{F}_4(k, \ell)$
  - this was made rigorous by a framework for *descent in buildings* recently
  - ... but there is no algebraic underpinning of this idea.

More disturbances: exotic pseudo-reductive groups

- CGP were classifying pseudo-reductive groups.
- Their standard construction describes most of them

$$\mathcal{G} = \mathrm{R}_{\ell/k} G \text{ (+ Cartan troubles).}$$

pseudo-reductive groups $\Longleftrightarrow$ semi-simple algebraic groups

- ... but they also found exotic groups!
- (and others actually)

More disturbances: exotic pseudo-reductive groups

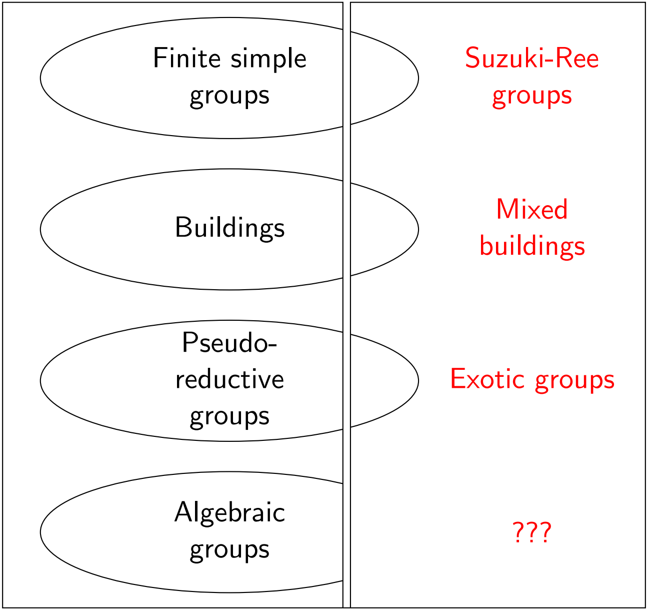
- Let k field of characteristic 2, 2, 3, 2 and G a simply connected k-group of type B_n, C_n, G_2, F_4.
- There is a very special isogeny between k-groups $\pi : G \to \bar{G}$ which factors the relative Frobenius $G \to G^{(p)}$.
- Let k'/k be a finite extension such that $k'^p \subseteq k$ and consider

$$f = \mathrm{R}_{k'/k}\pi_{k'} : \mathrm{R}_{k'/k}G_{k'} \to \mathrm{R}_{k'/k}\bar{G}_{k'}$$

- There is also a map $\bar{G} \to \mathrm{R}_{k'/k}\bar{G}_{k'}$ and the exotic group $\mathscr{G}$ is f^{-1} of its image.

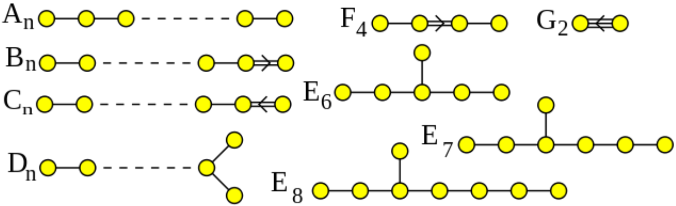
In short: $\mathscr{G}$ arises by thickening half of the group from k to k'.

Summary of disturbances



5 — The dark side of the moon

Near side: Combinatorics of root systems

- Relevant dynkin diagrams are always: B_n, C_n, F_4, G_2
- 

Far side: Mathematics in positive characteristic

- $p \mid \binom{p}{i}$ for $0 < i < p$, thus $(a+b)^p = a^p + b^p$
- There is a Frobenius (absolute and relative)
- The are imperfect fields and inseparable field extensions

We focus on this aspect.

Half a menhir

We must often thing bigger to solve our problems in mathematics. Consider constructing $\mathbb{Q}$ from $\mathbb{Z}$:

- Obelix only knows integers $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$.
- Obelix wants to solve $2x = 1$.
- You know a construction $\mathbb{Q} := (\mathbb{Z} \times \mathbb{Z})/\sim$.
- You 'solve' the Obelix-equation:

$$x = \frac{1}{2} := \{(1,2), (2,4), (3,6), (-1,-2), \dots\}$$

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How to see things bigger

General idea

The construction of a single 'new number' itself is not revealing; the importance comes from the observation that *as a whole* the new numbers have good properties.

- $\mathbb{N} \to \mathbb{Z}$ (the *Grothendieck group of a monoid*)
- $\mathbb{Z} \to \mathbb{Q}$ (the *fraction field of a ring*)
- $\mathbb{Q} \to \mathbb{R}$ (the *completion of a valued field*)
- $\mathbb{R} \to \mathbb{C}$ (the *algebraic closure* of a field)
- $(\mathbf{sch})_{/\mathbb{F}_p} \to (\mathbf{sch})_{/\mathbb{F}_{\sqrt{p}}}$ (the *twist of a category with end of Id.*)

How to see things bigger — in five easy steps

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- $(\mathbf{sch})_{/\mathbb{F}_p} \to (\mathbf{sch})_{/\mathbb{F}_{\sqrt{p}}}$ (the *twist of a category with end of Id.*)

General procedure to construct a new number.

1. Consider the set of old numbers (endowed with structure).
2. Use the structure to create another set of new numbers.
3. (Endow the new numbers with the desired structure.)
4. Provide a natural embedding of old numbers into new numbers.
5. We have now enlarged the old numbers and we can study what we have.

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- $\mathbb{Q} \to \mathbb{R}$ (the *completion of a valued field*)
- $\mathbb{R} \to \mathbb{C}$ (the *algebraic closure* of a field)
- $(\mathbf{sch})_{/\mathbb{F}_p} \to (\mathbf{sch})_{/\mathbb{F}_{\sqrt{p}}}$ (the *twist of a category with end of Id.*)

General procedure to construct a new scheme.

1. Consider the category of schemes $/\mathbb{F}_p$ (endowed with Frobenius).
2. Use the Frobenius to create a category of schemes $/\mathbb{F}_{\sqrt{p}}$.
3. (Endow the schemes $/\mathbb{F}_{\sqrt{p}}$ with the desired Frobenius.)
4. Provide a natural embedding of schemes $/\mathbb{F}_p$ into schemes $/\mathbb{F}_{\sqrt{p}}$.
5. We have now enlarged the schemes $/\mathbb{F}_p$ and we can study what we have.

6 — Twisting a category

The structure on $(\mathbf{Sch})_{/\mathbb{F}_p}$

1. Consider the category of schemes $/\mathbb{F}_p$ (endowed with Frobenius).

The objects X in the category $\mathscr{C} = (\mathbf{Sch})_{/\mathbb{F}_p}$ have $F_X : X \to X$ such that for all arrows u

$$\begin{array}{ccc} X & \xrightarrow{F_X} & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow[F_Y]{} & Y. \end{array}$$

I.e. F is an endomorphism of the identity functor

$$F : \mathrm{id}_{\mathscr{C}} \to \mathrm{id}_{\mathscr{C}}.$$

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$$\begin{array}{ccc} \mathrm{id}_{\mathscr{C}}(X) & \xrightarrow{F_X} & \mathrm{id}_{\mathscr{C}}(X) \\ \downarrow{\scriptstyle \mathrm{id}_{\mathscr{C}}(u)} & & \downarrow{\scriptstyle \mathrm{id}_{\mathscr{C}}(u)} \\ \mathrm{id}_{\mathscr{C}}(Y) & \xrightarrow[F_Y]{} & \mathrm{id}_{\mathscr{C}}(Y). \end{array}$$

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Twisting a category

2. Use the Frobenius to create a category of schemes $/\mathbb{F}_{\sqrt{p}}$.

Definitions

$(\mathscr{C}, F)$ is a category with endomorphism of the identity functor.
We define $t\mathscr{C}$:

- $t\mathscr{C}$: objects $(X, \Phi_X : X \to X)$ such that $\Phi_X \circ \Phi_X = F_X$
- $t\mathscr{C}$: arrows $u : (X, \Phi_X) \to (Y, \Phi_Y)$ are the arrows $u : X \to Y$ such that

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ \Phi_X \downarrow & & \downarrow \Phi_Y \\ X & \xrightarrow{u} & Y \end{array}$$

Twisting a category

2. Use the Frobenius to create a category of schemes $/\mathbb{F}_{\sqrt{p}}$.

Example

- Fields (k, θ) with $\theta^2 = \text{fr}$ as for twisted groups.
- Algebraic groups (G, π) with $\pi : G \to G$ such that $\pi^2 = \mathrm{Fr}_G$.

Twisting a category

3. (Endow the schemes $/\mathbb{F}_{\sqrt{p}}$ with the desired endomorphism of the identity functor.)

(Can be done with Φ_X itself.)

Twisting a category

4. Provide a natural embedding of schemes $/\mathbb{F}_p$ into schemes $/\mathbb{F}_{\sqrt{p}}$.

$$\begin{aligned} \mathrm{t} : \mathscr{C} &\longrightarrow t\mathscr{C} \\ X &\rightsquigarrow t(X) = (X \sqcup X, \tau \circ (F_X \sqcup \mathrm{id}_X)) \end{aligned}$$

The functor t:

- is faithful (so really an embedding) ...
- but neither full ...
- nor essentially surjective. (So lots of space!)

In particular:

$$\begin{array}{ccc} X & & t(X) \\ \downarrow & \rightsquigarrow & \downarrow \\ \operatorname{Spec} \mathbb{F}_p & & \mathrm{t}(\operatorname{Spec} \mathbb{F}_p) \end{array}$$

The embedding

5. We have now enlarged the schemes $/\mathbb{F}_p$ and we can study what we have.

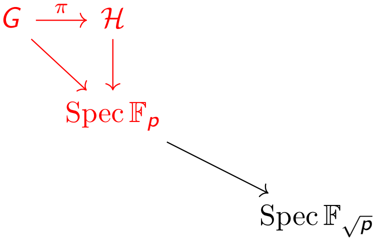
Embedding $\mathscr{C}$ into $t\mathscr{C}$:

$$\begin{array}{ccc} G & \xrightarrow{\pi} & \mathcal{H} \\ & \searrow & \downarrow \\ & & \operatorname{Spec} \mathbb{F}_p \end{array}$$

The embedding

5. We have now enlarged the schemes $/\mathbb{F}_p$ and we can study what we have.

Embedding $\mathscr{C}$ into $t\mathscr{C}$:



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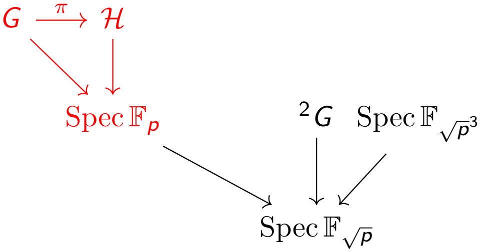
Embedding $\mathscr{C}$ into $t\mathscr{C}$:



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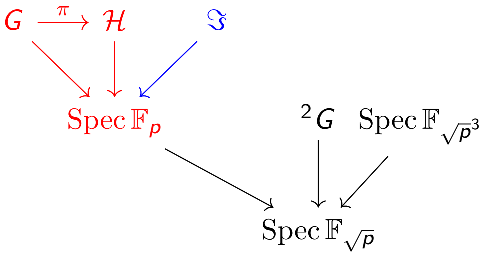

Mixed schemes = twisted schemes $/\mathbb{F}_p$

Invisible schemes $\Im$!? There are more schemes $/\mathbb{F}_p$ than we thought!

On mixing

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Mixed schemes = twisted schemes $/\mathbb{F}_p$

Invisible schemes $\Im$!? There are more schemes $/\mathbb{F}_p$ than we thought!

Example

- Consider fields k, ℓ such that $\ell^p \subsetneq k \subsetneq \ell$ then

$$(k \times \ell, (u, v) \mapsto (v^p, u))$$

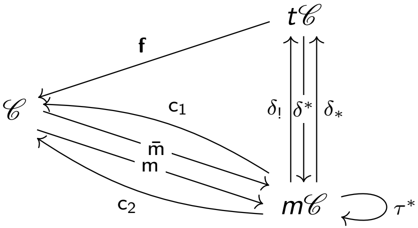
  is such an *invisible* field
- Consider an algebraic group G with factorization

$$G \to H \to G^{(p)}$$

  of the *relative Frobenius* then this gives rise to an *invisible* group.

Twisting a category

5. We have now enlarged the schemes $/\mathbb{F}_p$ and we can study what we have.



$$\delta_! \dashv \delta^* \dashv \delta_*$$

$$c_1 \dashv m \dashv c_2 \dashv \bar{m} \dashv c_1$$

7 — Theorems

Theorems

Theorems.

1. Suzuki-Ree groups are algebraic groups $/\mathbb{F}_{\sqrt{p}}$.
2. Mixed groups are *invisible* algebraic groups $/\mathbb{F}_p$.
3. Exotic groups arise from Weil restrictions coming from invisible fields.

(For a good notion of "are" and "arise".)

Theorems

Easy proof of (1).

Let $\tilde{k} = (k, \sigma)$ be a field with Tits endomorphism and $\tilde{G} = (G, \pi)$ an algebraic group over $\mathbb{F}_p$ with $\pi \circ \pi = F_G$.
Then $\tilde{G}(\tilde{k}) = \{u \in G(k) \mid \pi \circ u = u \circ \sigma\}$

$$\begin{array}{ccc} \text{Spec } k & \xrightarrow{\sigma} & \text{Spec } k \\ \Big\downarrow u & & \Big\downarrow u \\ G & \xrightarrow{\quad\pi\quad} & G \end{array}$$

For perfect k, this means that u is a fixed point of $u \mapsto \pi \circ u \circ \sigma^{-1}$. □

(It also provides a definition for the imperfect case)

Theorems

Plausibility check for (2).

Let $\tilde{X} = (X, Y, \alpha, \beta)$ be a mixed scheme over a mixed base $\tilde{S} = (\text{Spec } k, \text{Spec } \ell, \kappa, \lambda)$. There is a map

$$\tilde{X}(\tilde{S}) \to X(k) : (u, v) \mapsto u$$

$$\begin{array}{ccccccc} X & \underset{\beta}{\overset{\alpha}{\rightleftarrows}} & Y & & X \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} & \mapsto & \downarrow{\scriptstyle u} \\ K & \underset{\lambda}{\overset{\kappa}{\rightleftarrows}} & L & & K \end{array}$$

This map is *injective*:

$$u = u' \implies u \circ \beta = u' \circ \beta \implies \lambda \circ v = \lambda \circ v' \implies v = v'$$

This corresponds to the embedding $\tilde{X}(k, \ell) \subset X(k)$. □

Proof of (3)?

Not so simple! Relies on this proposition:
Consider a mixed object $\tilde{S}$ together with its morphism $f : \tilde{S} \to S = \mathrm{mc}_1(\tilde{S})$. Let $\tilde{X} = (X, X', \phi, \psi)$ be an $\tilde{S}$-object and assume $\beta_*\beta^*X$ and β_*X' exist. If we define

$$f_*(\tilde{X}) = \left(X, X \underset{\beta_*\beta^*X}{\times} \beta_*X', \pi, p_1\right),$$

then for all S-objects $\tilde{T}$:

$$\hom_S(\tilde{T}, f_*\tilde{X}) \simeq \hom_{\tilde{S}}(f^*\tilde{T}, \tilde{X})$$

where the maps are defined by ...

The point then is that this weird object $X \underset{\beta_*\beta^*X}{\times} \beta_*X'$ is precisely the CGP-construction but in a better notation. □

Further reading

- "Twisting and Mixing"
  https://arxiv.org/abs/1703.03794 (preprint)
- The author's PhD thesis (forthcoming)