Moufang Quadrangles

A unifying algebraic structure, and some results on exceptional quadrangles

Promotors: Hendrik Van Maldeghem Richard M. Weiss
Generalized Quadrangles

- Abstract structure of points and lines.
Abstract structure of points and lines.
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→ exactly one
Abstract structure of points and lines.

- Exactly one

\[ \geq 2 \]

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Generalized Quadrangles

- Abstract structure of points and lines.

- \( \rightarrow \text{exactly one} \)

- \( \geq 3 \)

- Extra condition: thick generalized quadrangle.
Bipartite graph; the partitions are called **points** and **lines**.

**Diameter** (max. distance between 2 vertices) $= 4$.
**Girth** (length of a smallest circuit) $= 8$.

Every vertex of such a graph has valency $\geq 2$.

If the valency at every vertex is $\geq 3$ $\Rightarrow$ thick generalized quadrangle.
Bipartite graph; the partitions are called **points** and **lines**.

Diameter (max. distance between 2 vertices) $= n$.
Girth (length of a smallest circuit) $= 2n$.

Every vertex of such a graph has valency $\geq 2$.

If the valency at every vertex is $\geq 3$  
$\Rightarrow$ thick generalized $n$-gon.
Smallest Thick Quadrangle
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An automorphism describes a “symmetry” of an object.
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All automorphisms together form an automorphism group.
Moufang Quadrangles

An ordinary quadrangle (□️) inside a generalized quadrangle is called an apartment.
Moufang Quadrangles

- An ordinary quadrangle (□) inside a generalized quadrangle is called an apartment.
- A root is half an apartment.

or

“dual”
Moufang Quadrangles

- An ordinary quadrangle (□) inside a generalized quadrangle is called an **apartment**.
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+ “dual” property

→ Moufang quadrangle
Fix a root, say $\alpha$. 
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$\rightarrow \text{root group } U_\alpha$
Fix a root, say $\alpha$.

$\rightarrow$ root group $U_\alpha$

Fix an apartment, say $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7\}$. 
Fix a root, say $\alpha$.

$\rightarrow$ root group $U_\alpha$

Fix an apartment, say $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

$\rightarrow$ root $(1, 2, 3, 4, 5)$
$\rightarrow$ root $(2, 3, 4, 5, 6)$
$\rightarrow$ root $(3, 4, 5, 6, 7)$
$\rightarrow$ root $(4, 5, 6, 7, 0)$
Fix a root, say $\alpha$.

$\rightarrow$ root group $U_\alpha$

Fix an apartment, say $\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7\}$.

$\rightarrow$ root $(1, 2, 3, 4, 5) \rightarrow$ group $U_1$

$\rightarrow$ root $(2, 3, 4, 5, 6) \rightarrow$ group $U_2$

$\rightarrow$ root $(3, 4, 5, 6, 7) \rightarrow$ group $U_3$

$\rightarrow$ root $(4, 5, 6, 7, 0) \rightarrow$ group $U_4$
Consider the root groups $U_1, U_2, U_3$ and $U_4$. 
Consider the root groups $U_1$, $U_2$, $U_3$ and $U_4$.

Consider the group $U_+ := \langle U_1, U_2, U_3, U_4 \rangle$. 

Theorem. (J. Tits)
The Moufang quadrangle is completely determined by $(U_+, U_1, U_2, U_3, U_4)$.

Corollary. (J. Tits)
The Moufang quadrangle is completely determined by $U_1$, $U_2$, $U_3$ and $U_4$ and the commutator relations between any two of these groups.
Commutator Relations

Consider the root groups $U_1, U_2, U_3$ and $U_4$.
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The Moufang quadrangle is completely determined by $U_1, U_2, U_3$ and $U_4$ and the commutator relations between any two of these groups.
Commutator Relations

Consider the root groups $U_1, \ldots, U_n$.

Consider the group $U_+ := \langle U_1, \ldots, U_n \rangle$.

**Theorem.** (J. Tits)

The Moufang $n$-gon is completely determined by $(U_+, U_1, \ldots, U_n)$.

**Corollary.** (J. Tits)

The Moufang $n$-gon is completely determined by $U_1, \ldots, U_n$ and the commutator relations between any two of these groups.
Parametrizing Structures

- Moufang triangles \((n = 3)\)
- Moufang quadrangles \((n = 4)\)
- Moufang hexagons \((n = 6)\)
- Moufang octagons \((n = 8)\)
Moufang triangles ($n = 3$)
→ alternative division rings [R. Moufang, 1933]

Moufang quadrangles ($n = 4$)

Moufang hexagons ($n = 6$)
→ hexagonal systems [J. Tits, late 60’s]

Moufang octagons ($n = 8$)
→ octagonal systems [J. Tits, middle 70’s]
Parametrizing Structures

- **Moufang triangles** \((n = 3)\)
  → alternative division rings [R. Moufang, 1933]

- **Moufang quadrangles** \((n = 4)\)
  → six different classes [J. Tits & R. Weiss, 2002]

- **Moufang hexagons** \((n = 6)\)
  → hexagonal systems [J. Tits, late 60’s]

- **Moufang octagons** \((n = 8)\)
  → octagonal systems [J. Tits, middle 70’s]
Parametrizing Structures

- Moufang triangles \((n = 3)\)
  → alternative division rings [R. Moufang, 1933]

- Moufang quadrangles \((n = 4)\)
  → quadrangular systems [2003]

- Moufang hexagons \((n = 6)\)
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- Moufang octagons \((n = 8)\)
  → octagonal systems [J. Tits, middle 70’s]
Quadrangular Systems

- abelian group \((V, +)\);
- (non-)abelian group \((W, \Box)\);
- “actions” \(V \cdot W = V\) and \(W \cdot V = W\);
- bi-additive map \(F : V \times V \rightarrow W\);
- bi-additive map \(H : W \times W \rightarrow V\);
- \(\epsilon \in V^*\) such that \(\omega \epsilon = \omega\);
- \(\delta \in W^*\) such that \(\nu \delta = \nu\);
- map on \(V^* : \nu \mapsto \nu^{-1} : \omega \nu \cdot \nu^{-1} = \omega\);
- map on \(W^* : \omega \mapsto \kappa(\omega) : \nu(\Box \omega) \cdot \kappa(\omega) = -\nu\);
- +16 other axioms.
Quadrangular Systems: Axioms

\(w \epsilon = w.\)

\(v \delta = v.\)

\((w_1 \boxplus w_2)v = w_1 v \boxplus w_2 v.\)

\((v_1 + v_2)w = v_1 w + v_2 w.\)

\(w(-\epsilon)v = w(-v).\)

\(v(w(-\epsilon)) = vw.\)

\(\text{Im}(F) \subseteq \text{Rad}(H).\)

\([w_1, w_2 v] \boxplus = F(H(w_2, w_1), v).\)

\(\delta \in \text{Rad}(H).\)

If \(\text{Rad}(F) \neq 0, \) then \(\epsilon \in \text{Rad}(F).\)

\(w(v_1 + v_2) = wv_1 \boxplus wv_2 \boxplus F(v_2 w, v_1).\)

\(v(w_1 \boxplus w_2) = vw_1 + vw_2 + H(w_2, w_1 v).\)

\((v^{-1})^{-1} = v.\)

\(\kappa(\boxplus \kappa(\boxplus w)) = w(-\epsilon).\)

\(wvv^{-1} = w.\)

\(v^{-1}(vv) = -v(\boxplus v).\)

\(F(v_1^{-1}, \overline{v_2})v_1 = F(v_1, v_2).\)

\(v \kappa(w)(\boxplus w) = -v.\)

\(w(v \kappa(w)) = \kappa(w)v.\)

\(H(\kappa(w_1), w_2)w_1 = H(w_1, w_2).\)
Commutator Relations

\[ U_1 \cong U_3 \cong (W, \Box) ; \]
\[ U_2 \cong U_4 \cong (V, +) ; \]
Commutator Relations

\[ U_1 \cong U_3 \cong (W, \Box) ; \]
\[ U_2 \cong U_4 \cong (V, +) ; \]
\[ [U_1, U_2] = [U_2, U_3] = [U_3, U_4] = 1 ; \]
\[ [x_1(w_1), x_3(w_2)^{-1}] = x_2(H(w_1, w_2)) ; \]
\[ [x_2(v_1), x_4(v_2)^{-1}] = x_3(F(v_1, v_2)) ; \]
\[ [x_1(w), x_4(v)^{-1}] = x_2(vw)x_3(wv) . \]
More Definitions

- $\Omega = (V, W, F, H, \epsilon, \delta)$: quadrangular system.
- $\Omega$ is **indifferent** if $F = 0$ and $H = 0$;
- $\Omega$ is **reduced** if $F \neq 0$ and $H = 0$;
- $\Omega$ is **wide** if $F \neq 0$ and $H \neq 0$. 
More Definitions

- $\Omega = (V, W, F, H, \epsilon, \delta)$: quadrangular system.
- $\Omega$ is **indifferent** if $F = 0$ and $H = 0$; $\Omega$ is **reduced** if $F \neq 0$ and $H = 0$; $\Omega$ is **wide** if $F \neq 0$ and $H \neq 0$.
- Every **wide** quadrangular system is the **extension** of a **reduced** one.
More Definitions

- \( \Omega = (V, W, F, H, \epsilon, \delta) \): quadrangular system.
- \( \Omega \) is **indifferent** if \( F = 0 \) and \( H = 0 \); \( \Omega \) is **reduced** if \( F \neq 0 \) and \( H = 0 \); \( \Omega \) is **wide** if \( F \neq 0 \) and \( H \neq 0 \).
- Every **wide** quadrangular system is the **extension** of a **reduced** one.
- Let \( \Omega \) be reduced. Then \( \Omega \) is **normal**
  \( \iff \forall w_1, w_2, \ldots, w_i \in W : \exists w \in W : \epsilon w_1 w_2 \ldots w_i = \epsilon w. \)
Classification

- **Ω indifferent**
  \[ \Rightarrow \Omega \cong \Omega_D(K, K_0, L_0) : \text{indifferent type.} \]

- **Ω reduced but not normal**
  \[ \Rightarrow \Omega \cong \Omega_I(K, K_0, \sigma) : \text{proper involutory type.} \]

- **Ω reduced and normal**
  \[ \Rightarrow \Omega \cong \Omega_Q(K, V_0, q) : \text{quadratic form type.} \]

- **Ω extension of a proper Ω_I(K, K_0, \sigma)**
  \[ \Rightarrow \Omega \cong \Omega_P(K, K_0, \sigma, V_0, p) : \text{pseudo-quadratic form type.} \]
Classification

- $\Omega$ extension of an $\Omega_Q(K, V_0, q)$
  - $\text{Rad}(F) \neq 0$
    - $\Rightarrow \Omega \cong \Omega_F(K, V_0, q)$: exceptional type $F_4$.
  - $\text{Rad}(F) = 0$
    - $\Rightarrow d := \dim_K V_0 \in \{2, 4, 6, 8, 12\}$
    - $d \in \{2, 4\}$: pseudo-quadratic form type.
    - $d = 6$: exceptional type $E_6$.
    - $d = 8$: exceptional type $E_7$.
    - $d = 12$: exceptional type $E_8$. 
Automorphisms

\[ G \] = full automorphism group.

\[ G_y \] = subgroup of \[ G \] generated by all root groups.

\[ G_y \] is a normal subgroup of \[ G \], which is simple (except for three tiny cases).

Interesting problem: Examine \[ G = G_y \ ].
$G := \text{full automorphism group.}$
Automorphisms

- $G := \text{full automorphism group.}$

- $G^\dagger := \text{subgroup of } G \text{ generated by all root groups.}$
Automorphisms

- \( G := \) full automorphism group.

- \( G^\dagger := \) subgroup of \( G \) generated by all root groups.

- \( G^\dagger \) is a normal subgroup of \( G \), which is simple (except for three tiny cases).
Automorphisms

- $G :=$ full automorphism group.

- $G^\dagger :=$ subgroup of $G$ generated by all root groups.

- $G^\dagger$ is a **normal** subgroup of $G$, which is **simple** (except for three tiny cases).

- Interesting problem: Examine $G/G^\dagger$. 
Completely solved in Tits-Weiss for the cases $n = 3$ and $n = 8$. 
\( G/G^{\dagger} \) - Problem

- Completely solved in Tits-Weiss for the cases \( n = 3 \) and \( n = 8 \).

- Solved in Tits-Weiss for the cases \( n = 4 \) and \( n = 6 \), except:
  - Exceptional quadrangles of type \( E_6, E_7, E_8 \).
  - Exceptional quadrangles of type \( F_4 \).
  - Exceptional hexagons of type \( E_8 \).
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Solved in Tits-Weiss for the cases $n = 4$ and $n = 6$, except:

- Exceptional quadrangles of type $E_6$, $E_7$, $E_8$.
- Exceptional quadrangles of type $F_4$.
- Exceptional hexagons of type $E_8$. 
Quadrangles of Type $F_4$

\[ [x_1(x, y, t), x_4(u, v, s)] = x_2(U, V, S) \cdot x_3(X, Y, T) \text{ where} \]

\[ U = \alpha \cdot (\bar{x}v + \beta y\bar{v}) + tu; \]
\[ V = xu + \beta y\bar{u} + tv; \]
\[ S = \hat{q}(x, y, t)s + \alpha \cdot (x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha \cdot (xy\bar{v}^2 + x\bar{y}v^2)); \]
\[ X = y\bar{u}^2 + \alpha \bar{y}v^2 + sx; \]
\[ Y = \beta^{-2} \cdot (xu^2 + \alpha \bar{x}v^2) + sy; \]
\[ T = q(u, v, s)t + \alpha \cdot (\beta^{-1} \cdot (xu\bar{v} + \bar{x}u\bar{v}) + y\bar{u}d + \bar{y}uv). \]
Quadrangles of Type $F_4$

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\[
S = \hat{q}(x, y, t)s + \alpha \cdot (x\bar{y}u^2 + \bar{x}y\bar{u}^2 + \alpha \cdot (xy\bar{v}^2 + \bar{x}y\bar{v}^2));
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X = y\bar{u}^2 + \alpha \bar{y}v^2 + sx;
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T = q(u, v, s)t + \alpha \cdot (\beta^{-1} \cdot (xu\bar{v} + \bar{x}u\bar{v}) + y\bar{u}\bar{v} + \bar{y}uv).
\]

\[
[x_1(w), x_4(v)] = x_2(vw) \cdot x_3(wv).
\]
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$G_y : \cong \text{Aut}(X)$

$G_{h_c, \hat{z}} : \cong \text{SelfSim}(y)$

$G_{h_c, \hat{z}} : \cong G_{y}$

$X : \cong \text{root groups}$
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where $\chi_c, \hat{\chi}_z$ are self-similarities induced by reflections in quadratic spaces.
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$$G/G^\dagger \cong X/X^\dagger$$
Construction of $\Omega$ is based on two quadratic spaces $(K, V, q)$ and $(L, W, \hat{q})$, with $L < K$. 
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- $X_\ell := \{ \text{linear self-similarities} \}$.
- $X/X_\ell \cong A \leq \text{Aut}(K, L)$. 
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$X_\ell := \{ \text{linear self-similarities} \}$.

$X/X_\ell \cong A \leq \text{Aut}(K, L)$.

$X_\ell \cong X^\dagger$

- Determination of $G(q) := \text{group of multipliers of similitudes of } q$.
- Cartan-Dieudonné-type theorem for $q$.
- Restriction of the parameters of the self-similarity.
\[ G/G^\dagger \cong X/X^\dagger. \]
\[ X^\dagger \cong X_\ell. \]
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\[ X / X_\ell \cong A \leq \text{Aut}(K, L). \]

\[
\implies G / G^\dagger \cong A \leq \text{Aut}(K, L)
\]
To do ...

Try to use similar methods to solve the $G = G_y$-problem for the exceptional quadrangles of type $E_6$, $E_7$ and $E_8$.

Generalize the concept of algebraic groups to pairs of fields, to include the exceptional quadrangles of type $F_4$.

Try to find a deeper connection with Jordan algebras.

Applications?
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