Moufang sets arising from Moufang polygons of type E_6 and E_7

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June 30, 2006

Abstract

We explicitly show that Moufang quadrangles of type E_6 and E_7 have classical rank one residues with non-abelian root groups, by calculating the isomorphism between the exceptional and the pseudo-quadratic rank one groups. This fact is clear from the diagrams which describe these Moufang quadrangles, but for certain purposes it might be interesting to have an explicit isomorphism available. This note is not intended for publication as it is.

1 Moufang sets arising from Moufang polygons of type E_6, E_7 and E_8

We will make very intensive use of [2], in particular of the description of the Moufang quadrangles of type E_k as in Chapters 13 and 16, and we refer the reader to [2] for the definition of all the maps which occur in the sequel.

We first explicitly describe the Moufang sets arising from Moufang polygons of type E_6, E_7 and E_8 in terms of one group $(U, +)$ and a permutation τ, as in [1]. This is easy to obtain from the explicit formulas in [2, (32.10)]. The group U is simply the group S, and the permutation τ can be obtained as $\tau = \kappa \circ \mathrm{inv} = \mathrm{inv} \circ \lambda$, where $\mu(x_1(z)) = x_5(\kappa(z))x_1(z)x_5(\lambda(z))$, and where $\mathrm{inv}(z) := -z$, for all $z \in S^*$. We get that $S = X \times k$ as a set, where the (non-commutative) addition in S is given by

$$(a, t) + (b, s) = (a + b, t + s + g(a, b))$$

for all $(a, t), (b, s) \in S$; the map τ is given by

$$\tau(a, t) = \Big(\frac{a \cdot \overline{(\pi(a) + t\epsilon)}}{q(\pi(a) + t\epsilon)}, \frac{-t + g(a, a)}{q(\pi(a) + t\epsilon)} \Big) \tag{1.1}$$

for all $(a, t) \in S^*$.

2 Moufang quadrangles of type E_6

We will now concentrate on the Moufang quadrangles of type E_6. Let

$$a = t_1 * v_1 + t_2 * v_2 + t_3 * v_3 + t_{23} * v_{23}$$

be an arbitrary element in X, where $t_1, t_2, t_3, t_{23} \in E$, and let

$$\Pi(a) := \tilde{Q}_1(a)\gamma - \tilde{Q}_2(a)\gamma^\sigma ,$$

where $\tilde{Q}_1$ and $\tilde{Q}_2$ are as in [3, Section 4]. In terms of the coordinates t_1, t_2, t_3, t_{23}, we get that

$$\Pi(a) = t_1\gamma t_1^\sigma - s_2 \cdot t_2\gamma^\sigma t_2^\sigma - s_3 \cdot t_3\gamma^\sigma t_3^\sigma + s_{23} \cdot t_{23}\gamma t_{23}^\sigma .$$

So Π is a pseudoquadratic form over E, with involution σ and with $E_0 = k$. By [3, Theorem 4.2], we have $a\pi(a) = \Pi(a) * a$ for all $a \in X$, so if $\Pi(a) \in k$, then $a\pi(a) \in ak$, but then $\pi(a) \in k\epsilon$, which can only happen if $a = 0$ by [2, (13.49)]. Hence Π is anisotropic.

Let $H : X \times X \to E$ be the corresponding skew-hermitian form, and let

$$G(a, b) := \Pi(a + b) - \Pi(a) - \Pi(b) - H(b, a)$$

for all $a, b \in X$. Observe that G is a map from $X \times X$ to k. We will show that G coincides with g. Indeed, let $a = t_1 * v_1 + t_2 * v_2 + t_3 * v_3 + t_{23} * v_{23}$ and $b = u_1 * v_1 + u_2 * v_2 + u_3 * v_3 + u_{23} * v_{23}$; then

$$\begin{aligned} G(a, b) = t_1\gamma u_1^\sigma - s_2 \cdot t_2\gamma^\sigma u_2^\sigma - s_3 \cdot t_3\gamma^\sigma u_3^\sigma + s_{23} \cdot t_{23}\gamma u_{23}^\sigma \\ + u_1\gamma t_1^\sigma - s_2 \cdot u_2\gamma^\sigma t_2^\sigma - s_3 \cdot u_3\gamma^\sigma t_3^\sigma + s_{23} \cdot u_{23}\gamma t_{23}^\sigma \\ - u_1\rho t_1^\sigma + s_2 \cdot u_2\rho^\sigma t_2^\sigma + s_3 \cdot u_3\rho^\sigma t_3^\sigma - s_{23} \cdot u_{23}\rho t_{23}^\sigma \\ = T\big(t_1\gamma u_1^\sigma - s_2 \cdot t_2\gamma^\sigma u_2^\sigma - s_3 \cdot t_3\gamma^\sigma u_3^\sigma + s_{23} \cdot t_{23}\gamma u_{23}^\sigma\big) . \end{aligned}$$

Since both G and g are bi-additive, and satisfy $G(t * a, t * b) = N(t)G(a, b)$ and $g(t * a, t * b) = N(t)g(a, b)$ by [3, Lemma 3.10], it suffices to check that $G(a, b) = g(a, b)$ for $a \in \{v_1, v_2, v_3, v_{23}\}$ and $b \in E*v_1 \cup E*v_2 \cup E*v_3 \cup E*v_{23}$. Recall that $g(a, b) = f(h(b, a), \delta)$ for all $a, b \in X$, where $\delta = \epsilon/2$ if $\mathrm{char}(k) \neq 2$ and $\delta = \eta/\rho = \gamma\epsilon/\rho$ if $\mathrm{char}(k) = 2$. In any case, if $a \in E*v_I$ and $b \in E*v_J$ for different $I, J \in \{\{1\}, \{2\}, \{3\}, \{23\}\}$, then $g(a, b) = 0$. It only remains to check that $g(v_I, t * v_I) = G(v_I, t * v_I)$ for each $I \in \{\{1\}, \{2\}, \{3\}, \{23\}\}$ and each $t \in E$.

We have $g(v_1, t * v_1) = f(\rho t^\sigma \epsilon, \delta)$; when $\mathrm{char}(k) \neq 2$, this is equal to $f(\rho t^\sigma \epsilon, \epsilon/2) = T(\gamma t^\sigma) = G(v_1, t * v_1)$; when $\mathrm{char}(k) = 2$, this is equal to $f(\rho t^\sigma \epsilon, (\gamma/\rho)\epsilon) = T(\gamma t^\sigma) = G(v_1, t * v_1)$ as well.

Next, consider $g(v_2, t*v_2) = s_2 f(\rho^\sigma t\epsilon, \delta)$; when $\mathrm{char}(k) \neq 2$, this is equal to $-s_2 f(\rho t\epsilon, \epsilon/2) = -s_2 T(\gamma t) = -s_2 T(\gamma^\sigma t^\sigma) = G(v_2, t*v_2)$; when $\mathrm{char}(k) = 2$, this is equal to $s_2 f(\rho t\epsilon, (\gamma/\rho)\epsilon) = -s_2 T(\gamma t) = -s_2 T(\gamma^\sigma t^\sigma) = G(v_2, t*v_2)$.

The proof of the remaining two cases is completely similar. We conclude that $g = G$.

3 Moufang quadrangles of type E_7

We now turn our attention to the Moufang quadrangles of type E_7. Let

$$a = z_1 * v_1 + z_2 * v_2 + z_3 * v_3 + z_4 * v_4$$

be an arbitrary element in X, where $z_1, z_2, z_3, z_4 \in D$, and let

$$\Pi(a) := \tilde{Q}_1(a)\gamma - \tilde{Q}_2(a)\gamma^\sigma + e_2\rho P(a)^\sigma ,$$

where $\tilde{Q}_1$ and $\tilde{Q}_2$ and P are as in [3, Section 5]. In terms of the coordinates z_1, z_2, z_3, z_4, we get that

$$\Pi(a) = z_1\gamma z_1^\sigma - s_2 \cdot z_2\gamma^\sigma z_2^\sigma - s_3 \cdot z_3\gamma^\sigma z_3^\sigma - s_4 \cdot z_4\gamma^\sigma z_4^\sigma .$$

So Π is a pseudoquadratic form over D, with involution σ and with $D_0 = k$. By [3, Theorem 4.2], we have $a\pi(a) = \Pi(a) * a$ for all $a \in X$, so if $\Pi(a) \in k$, then $a\pi(a) \in ak$, but then $\pi(a) \in k\epsilon$, which can only happen if $a = 0$ by [2, (13.49)]. Hence Π is anisotropic.

As in the E_6 case, we let $H : X \times X \to E$ be the corresponding skew-hermitian form, and let

$$G(a, b) := \Pi(a + b) - \Pi(a) - \Pi(b) - H(b, a)$$

for all $a, b \in X$. It can be shown, in a completely similar way as in the E_6 case, that $G(a, b) = g(a, b)$ for all $a, b \in X$.

4 The isomorphism to Moufang sets of pseudo-quadratic form type

We now consider the two cases E_6 and E_7 together. In the E_6 case, we let $D = E$; in the E_7 case, we keep our meaning of D as in the previous paragraph. We will explicitly describe the Moufang set $\mathbb{M}(\tilde{U}, \tilde{\tau})$ corresponding to the pseudoquadratic form $\Pi : X \to D$. Let $\tilde{U} = X \times k$ as a set, with addition in $\tilde{U}$ given by

$$(a, t) + (b, s) = (a + b, t + s + G(a, b))$$

for all $(a, t), (b, s) \in \tilde{U}$. Using [2, (32.9)] and applying the isomorphism $T \to \tilde{U} : (a, t) \mapsto (a, t - \Pi(a))$, we obtain that the map $\tilde{\tau}$ is given by

$$\tilde{\tau}(a, t) = \Big((\Pi(a) + t)^{-1} * a, \frac{-t + T(\Pi(a))}{N(\Pi(a) + t)} \Big) \qquad (4.1)$$

for all $(a, t) \in \tilde{U}^*$.

Lemma 4.1. *For all $(a, t) \in S$, we have*

(i) $T(\Pi(a)) = g(a,a)$;

(ii) $N(\Pi(a)+t) = q(\pi(a)+t\epsilon)$;

(iii) $(\Pi(a)+t) * a = a \cdot (\pi(a)+t\epsilon)$;

(iv) $(\Pi(a)+t)^\sigma * a = a \cdot \overline{(\pi(a)+t\epsilon)}$.

Proof. (i) It follows from the definition of G that $G(a,a) = 2\Pi(a) - H(a,a)$. By [2, (11.19)], however, $H(a,a) = \Pi(a) - \Pi(a)^\sigma$, and hence $G(a,a) = \Pi(a) + \Pi(a)^\sigma = T(\Pi(a))$. Since we have shown that $G = g$, this implies (i).

(ii) In the E_6 case, this follows from [3, Theorem 4.2] and a little bit of calculation. In the E_7 case, this is precisely [3, Theorem 5.12].

(iii) In the E_6 case, this is [3, Theorem 4.2]; in the E_7 case, this is [3, Theorem 5.3].

(iv) This follows from (i) and (iii) since $\pi(a) + \overline{\pi(a)} = g(a,a)$.

□

Theorem 4.2. *The Moufang sets* $\mathbb{M}(S,\tau)$ *and* $\mathbb{M}(\tilde{U},\tilde{\tau})$ *are isomorphic.*

Proof. This follows from the expressions (1.1), (4.1), and Lemma 4.1. □

References

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