Non-abelian Moufang sets of type $^2A_3$, C_3 and F_4

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Abstract

We explicitly calculate the Hua maps for the Moufang sets of type $^2A_3$, C_3 and F_4. We observe that there can be infinitely many different elements with the same Hua map. This note is not intended for publication as it is.

1 Description of the Moufang sets

Let k be an arbitrary commutative field, and let A be either a separable quadratic extension field of k, a quaternion division algebra over k, or an octonion division algebra over k. Let σ be the standard involution of A/k, and let $N(a) := aa^\sigma$ and $T(a) := a + a^\sigma$ (for all $a \in A$) be the norm map and the trace map of A/k, respectively. Let

$$U := \{(a, b) \in A \times A \mid N(a) + T(b) = 0\} .$$

Then we can make U into a (non-abelian) group by defining the group "addition"

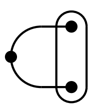
$$(a, b) + (c, d) := (a + c, b + d - c^\sigma a)$$

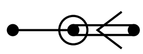
for all $(a, b), (c, d) \in U$; it is easily checked that this is indeed a group, with neutral element $(0, 0)$ and with the inverse given by $-(a, b) = (-a, b^\sigma)$. Now we define a permutation τ on U^*, by setting

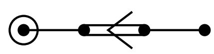
$$(a, b)\tau = (-ab^{-1}, b^{-1})$$

for all $(a, b) \in U^*$. Then $\mathbb{M}(U, \tau)$ is a Moufang set.

These examples all arise from algebraic groups of relative rank one. Depending on whether A is a quadratic extension, a quaternion division algebra or an octonion division algebra, the corresponding indices are as follows.

$^2A_3$ 

C_3 

F_4 

Remark 1.1. When $k = \mathsf{GF}(2)$ and $A = \mathsf{GF}(4)$, this gives the smallest example of a Moufang set with non-abelian root groups. It has $U \cong Q_8$, and hence $|X| = 9$, and $G \cong \mathsf{PSU}_2(2)$.

2 Description of the Moufang sets

We now explicitly compute the Hua maps. Recall that

$$h_x = \tau \cdot \alpha_x \cdot \tau^{-1} \cdot \alpha_{-(x\tau^{-1})} \cdot \tau \cdot \alpha_{-(-(x\tau^{-1}))\tau}$$

for all $x \in U^*$. Note that in our case, $\tau^2 = 1$ and hence $\tau^{-1} = \tau$. Let $x := (a, b) \in U^*$ be arbitrary. Then

$$\begin{aligned} -(x\tau^{-1}) &= (ab^{-1}, b^{-\sigma}) \,; \\ -(-(x\tau^{-1}))\tau &= (ab^{-1}b^{\sigma}, b) \,. \end{aligned}$$

We get

$$(c, d)\tau \cdot \alpha_x = (a - cd^{-1}, b + d^{-1} + a^{\sigma} \cdot cd^{-1})$$

and hence

$$(c, d)\tau \cdot \alpha_x \cdot \tau^{-1} = \big((cd^{-1} - a)(b + d^{-1} + a^{\sigma} \cdot cd^{-1}), (b + d^{-1} + a^{\sigma} \cdot cd^{-1})^{-1}\big)$$

for all $(c, d) \in U^*$. Now let

$$\begin{aligned} A &:= (cd^{-1} - a)(b + d^{-1} + a^{\sigma} \cdot cd^{-1}) + ab^{-1} \,; \\ B &:= (b + d^{-1} + a^{\sigma} \cdot cd^{-1})^{-1} + b^{-\sigma} - (ab^{-1})^{\sigma} \cdot (cd^{-1} - a)(b + d^{-1} + a^{\sigma} \cdot cd^{-1})^{-1} \,. \end{aligned}$$

We finally get

$$(c, d)h_x = (-AB^{-1} + ab^{-1}b^{\sigma}, B^{-1} + b + (ab^{-1}b^{\sigma})^{\sigma} \cdot AB^{-1}) \qquad (*)$$

for all $(c, d) \in U^*$.

We will from now on assume that A is a division ring (i.e. we exclude the case where A is an octonion division algebra). Then

$$\begin{aligned} B^{-1} &= (b + d^{-1} + a^{\sigma} \cdot cd^{-1}) \cdot db^{\sigma} = bdb^{\sigma} + b^{\sigma} + a^{\sigma}cb^{\sigma} \,, \\ AB^{-1} &= cb^{\sigma} + ab^{-1}b^{\sigma} + ab^{-1}a^{\sigma}cb^{\sigma} \,, \end{aligned}$$

and we get, using the fact that $aa^{\sigma} + b + b^{\sigma} = 0$ and keeping in mind that $aa^{\sigma} = aa^{\sigma} \in k = Z(A)$, that

$$(c, d)h_x = (ab^{-1}b^{\sigma}a^{-1}cb^{\sigma}, bdb^{\sigma}) \qquad (**)$$

for all $(c, d) \in U^*$.

Note that, if A is a quadratic extension of k, then it is commutative, and hence this formula simplifies to

$$(c,d)h_x = (b^{-1}(b^\sigma)^2 \cdot c, bb^\sigma \cdot d) \tag{†}$$

for all $(c,d) \in U^*$.

Remark 2.1. Formula (†) is independent of a, and hence there are many elements $x \in U^*$ which have the same Hua map h_x, namely all $x = (a,b)$ with fixed b and with any a such that $N(a) = -T(b)$. This is in great contrast with the case of special Moufang sets, where $h_x = h_y$ implies $x = \pm y$.

Remark 2.2. When A is an octonion division algebra, I expect a simplification of equation $(*)$ to a formula similar to equation $(**)$ above, but I haven't taken the time yet to do the (more involved) calculations.

References

[1] T. De Medts and R. M. Weiss, Moufang sets and Jordan division algebras, *Math. Ann.* **335** (2006), no. 2, 415–433.