The isomorphism problem for Moufang sets of type ${}^1D_4^{(2)}$ and the Bartels invariant

Tom De Medts*

Early 2007, with minor updates in 2017

Abstract

The algebraic groups of type ${}^1D_4^{(2)}$ are classified by triples of quaternion algebras whose sum is trivial in the Brauer group. We explain on an elementary level why this is true, and in the case where the characteristic of the underlying field is not 2, we relate it to a cohomological invariant which was introduced by Bartels.

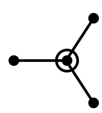
We then proceed to show that the Moufang sets (i.e., the buildings corresponding to these groups) contain enough information to recover the three quaternion algebras and the three corresponding skew-hermitian forms, thereby solving the "isomorphism problem" for this class of Moufang sets.

These notes date back to early 2007 but have been slightly updated in 2017 before making available online. They are not intended for publication in their current form. (The paper has been rejected by 3 different people in the past, which proves that it is not worth publishing; the main objection was that it is too elementary.)

1 Introduction

The algebraic groups of type D_4 are very intriguing. On the one hand, they belong to a classical family, but on the other hand, they often behave so differently that it is not exaggerated to include them in the list of exceptional groups. This is of course due to the peculiar symmetry in their diagram.

In this paper, we study the absolutely simple isotropic algebraic groups of type ${}^1D_4^{(2)}$ and relative rank one, i.e. those with Tits index



*The author is a Postdoctoral Fellow of the Research Foundation - Flanders (Belgium) (F.W.O.-Vlaanderen).

and not involving a diagram symmetry (i.e. of inner type). Of course, these groups are well understood, but we will focus on a peculiar aspect arising from the symmetry in the diagram. The anisotropic kernel G_0 of such a group is of type $A_1 \times A_1 \times A_1$, and hence $G_0 = SL_1(Q_1) \times SL_1(Q_2) \times SL_1(Q_3)$ for some quaternion division algebras Q_1, Q_2 and Q_3. It follows, in fact, from the theory of algebraic groups (see for example [12, Satz IV.6.3.1]) that it is necessary and sufficient for such a group to exist, that $[Q_1] + [Q_2] + [Q_3] = 0$ in the Brauer group $\mathrm{Br}(k)$.

However, the theory of algebraic groups does explain directly how these three quaternion algebras can be obtained from the skew-hermitian form which describes such a group, nor does it explain on an algebraic level why we should be unable to distinguish between these three quaternion algebras if we only know the algebraic group.

The first goal of this paper is precisely to understand this on a more elementary algebraic level, and this is what we aim to do in section 2. It turns out that, starting from one particular skew-hermitian form over one of the quaternion division algebras, the two others can be obtained from a certain cohomological invariant which was introduced by Bartels [2]. In our specific case, however, this invariant can be constructed in a very elementary way (although it would be difficult to trace down that it is indeed an invariant).

The second (and perhaps main) goal of this paper is to solve the "isomorphism problem" for the corresponding *Moufang sets.* Moufang sets were introduced by Jacques Tits [16] precisely as an axiomatization of the absolutely simple algebraic groups of relative rank one. (They were introduced in the context of twin buildings, but this is not important for us.)

In our case of groups of type ${}^1D_4^{(2)}$, these Moufang sets can be described in an elementary way only using the skew-hermitian form which determines the group. The underlying vector space, which is an 8-dimensional k-vector space, has the structure of a vector space over each of these three quaternion algebras Q_i simultaneously, in such a way that these three multiplications *pairwise* commute; in some sense, V becomes a "left", "right" and "middle" vector space at the same time (where it would of course be better to think of these as the three directions in the Dynkin diagram).

In section 3, we show that the three quaternion algebras (and the three corresponding skew-hermitian forms) can indeed be recovered starting from the Moufang set, thereby solving the isomorphism problem for this class of Moufang sets. It is interesting to note that we unexpectedly rely on the theory of spreads in incidence geometry for the final step of this process.

We have separated the characteristic 2 case from the rest (see section 4) for two reasons. First, the technical details in this case would interfere too much with the general ideas; second, the Bartels invariant is only defined when the characteristic is not 2, so we cannot pursue this link —see, however, Remark 2.5 below— though our main result continues to hold.

Acknowledgment

I am very grateful to Bernhard Mühlherr, for proposing this problem to me, and for the many valuable discussions we have had on this matter. I would also like to thank Jan Van Geel, for making me aware of the existence of the Bartels invariant, and Eva Bayer-Fluckiger, for pointing out that this invariant is essentially already due to Tits. I also thank Norbert Knarr, for providing a reference for Proposition 3.3.

2 Structure of the rank one groups of type ${}^1D_4^{(2)}$

Let k be an arbitrary commutative field with $\mathrm{char}(k) \neq 2$. We start by describing the algebraic groups of absolute type ${}^1D_4^{(2)}$ and k-rank one and their corresponding Moufang sets. By [13], these groups are the groups $SU_4(Q, f)$, where Q is a quaternion division algebra over k, and f is a non-degenerate skew-hermitian form with trivial discriminant and index 1, relative to the standard involution σ of Q. Such a form f is uniquely determined by its anisotropic part h, which is a form of dimension 2 over Q with trivial discriminant.

So let X be a 2-dimensional right vector space over Q, let σ be the standard involution of Q, and let N and T be the norm and the trace from Q to k, respectively, i.e. $N(t) = t^\sigma t$ and $T(t) = t^\sigma + t$ for all $t \in Q$. Let h be an anisotropic skew-hermitian form from $X \times X$ to Q. We denote the pseudo-quadratic form corresponding to h by π, i.e. $\pi(a) := h(a, a)/2$ for all $a \in X$. Observe that $T(\pi(a)) = 0$ for all $a \in X$. Let g be the map from $X \times X$ to k defined by

$$g(a, b) := \pi(a + b) - \pi(a) - \pi(b) - h(a, b) \tag{2.1}$$

for all $a, b \in X$; it is easily checked that

$$g(a, b) = T(h(b, a))/2 \tag{2.2}$$

for all $a, b \in X$.

2.1 The Moufang set

Definition 2.1. A *Moufang set* is a set X together with a collection of permutation groups $(U_x)_{x\in X}$, where each $U_x \leq \mathrm{Sym}(X)$ fixes x and acts sharply transitively on $X \setminus \{x\}$, such that $U_x^\psi = U_{x^\psi}$ for every $\psi \in G^+ :=$

$\langle U_y \mid y \in X\rangle$. The groups U_x are called the *root groups* of the Moufang set, and the group G^+ is called the *little projective group*.

This notion was introduced by J. Tits [16] as an axiomatization of some of the structure of the absolutely simple algebraic groups of relative rank one. More precisely, if $\mathbf{G}$ is such an algebraic group defined over k of k-rank 1, then the set X is the set of all minimal k-parabolics of $\mathbf{G}$, and each root group U_x is precisely the root subgroup of the k-parabolic subgroup x (which in this case coincides with the unipotent radical of the k-parabolic x); the action is given by conjugation. The group G^+ is precisely the group of k-rational points of the adjoint representation of $\mathbf{G}$.

Each Moufang set is completely determined by the structure of one (and hence all) of the root groups, together with one additional permutation τ of the non-trivial elements of this group, as we now explain.

Definition 2.2. Let $(U, +)$ be an arbitrary group (not necessarily abelian but nevertheless written additively), and let $\tau \in \mathrm{Sym}(U^*)$, where $U^* := U \setminus \{0\}$. Define a set $X := U \cup \{\infty\}$, where ∞ is a new symbol. We extend τ to an element of $\mathrm{Sym}(X)$ by demanding that it exchanges the elements 0 and ∞.

Next, we define subgroups $U_x \leq \mathrm{Sym}(X)$ as follows. For each $a \in U$, let α_a be the permutation of X fixing ∞ and mapping each $x \in U$ to $x + a$. Then $U_\infty := \{\alpha_a \mid a \in U\}$ is a subgroup of $\mathrm{Sym}(X)$ isomorphic to U. Now define $U_0 := U_\infty^\tau$ (where we mean conjugation by τ inside $\mathrm{Sym}(X)$), and for each $a \in U^*$, define $U_a := U_0^{\alpha_a}$. Denote the resulting data $(X, (U_x)_{x\in X})$ by $\mathbb{M}(U, \tau)$.

Then $\mathbb{M}(U, \tau)$ is not always a Moufang set, but every Moufang set can be obtained in this fashion; see, for instance, [4, 3].

The Moufang set corresponding to h, i.e. corresponding to the group $SU_4(Q, f)$, is equal to $\mathbb{M}(U, \tau)$, where U is the (abstract) non-abelian group with underlying set $X \times k$ and with group "addition" given by

$$(a, s) + (b, t) = (a + b, s + t + g(a, b))$$

for all $(a, s), (b, t) \in U$, and where τ is the map from U^* to itself given by

$$\tau\colon (a, s) \mapsto (a(s + \pi(a))^{-1}, -sN(s + \pi(a))^{-1})$$

for all $(a, s) \in U^*$.

Remark 2.3. These formulas can be calculated from [17, (32.9)]; note that our group U is isomorphic to the group T in [17] under the isomorphism $U \to T\colon (a, s) \mapsto (a, s + \pi(a))$. More precisely, the permutation τ can be obtained as $\tau = \kappa \circ \mathrm{inv} = \mathrm{inv} \circ \lambda$, where $\mu(x_1(z)) = x_5(\kappa(z))x_1(z)x_5(\lambda(z))$, and where $\mathrm{inv}(z) := -z$, for all $z \in U^*$. Observe that we have used the fact that $T(\pi(a)) = 0$ to obtain the formula for τ.

2.2 The structure of h

We will first examine the structure of the skew-hermitian form h. By [9, Chapter 7, Theorem 6.3], (X, h) has a basis consisting of two pairwise orthogonal vectors, say u and v, and hence $X = uQ \perp vQ$. So there exist fixed elements $z, w \in Q$ such that

$$h(ux_1 + vy_1, ux_2 + vy_2) = x_1^\sigma z x_2 + y_1^\sigma w y_2 \tag{2.3}$$

for all $x_1, x_2, y_1, y_2 \in Q$; since h is skew-hermitian, we have $T(z) = T(w) = 0$. Since $\mathrm{disc}(h) = 1 \in k^*/(k^*)^2$, we have $N(z)N(w) \in (k^*)^2$ and therefore $N(z) = r^2 N(w)$ for some $r \in k^*$. It follows that $\langle 1, z\rangle$ and $\langle 1, w\rangle$ are isomorphic subfields of Q. Since any two isomorphic subfields of Q are conjugate in Q, we have $t^\sigma w t \in \langle 1, z\rangle$ for some $t \in Q$. But since $T(t^\sigma w t) = T(z) = 0$, this implies that $t^\sigma w t = sz$ for some $s \in k^*$. (We are grateful to Richard Weiss for this argument.) If we now replace the basis vector v by $v' = vt$, then we get from (2.3) that

$$\begin{aligned} h(ux_1 + v'y_1, ux_2 + v'y_2) &= x_1^\sigma z x_2 + y_1^\sigma t^\sigma w t y_2 \\ &= x_1^\sigma z x_2 + s y_1^\sigma z y_2 \end{aligned} \tag{2.4}$$

for all $x_1, x_2, y_1, y_2 \in Q$. Conversely, any skew-hermitian form h of the form (2.4) has trivial discriminant.

Now let $E := \langle 1, z\rangle$; then E is a quadratic subfield of Q. Following the notation in [17], we write $Q = (E/k, \delta)$ where $\delta \in k^*$, i.e. $Q = E \oplus eE$, where $ae = ea^\sigma$ for all $a \in E$ and $e^2 = \delta$.

Lemma 2.4. *Let h be an arbitrary 2-dimensional skew-hermitian form over a quaternion division algebra Q with trivial discriminant, and write*

$$h(ux_1 + vy_1, ux_2 + vy_2) = x_1^\sigma z x_2 + s y_1^\sigma z y_2$$

for all $x_1, x_2, y_1, y_2 \in Q$. Let $Q = E \oplus eE$ as above. Then h is anisotropic if and only if

$$s \notin -N(E) \cup N(eE)\,.$$

Proof. First assume that h is anisotropic. If we would have $s = -N(t)$ for some $t \in E$, then we would have $2\pi(ut + v') = t^\sigma z t - N(t) z = 0$, contradicting the fact that h is anisotropic. If we would have that $s = N(et)$ for some $t \in E$, then we would have $2\pi(u(et) + v') = -etzet + N(et)z = 0$, again contradicting the fact that h is anisotropic.

Conversely, assume that h is isotropic. Then there exist elements $x, y \in Q$ such that $\pi(ux + vy) = 0$. (Observe that $\pi(ux + vy)$ can never be an element of k^* since its trace is 0.) Hence $x^\sigma z x = -s y^\sigma z y$. Let $w := xy^{-1}$, then this implies that

$$w^\sigma z w = -sz\,. \tag{2.5}$$

Taking norms, we get that $N(w)^2N(z) = s^2N(z)$ and hence $s = \pm N(w)$.

If $s = -N(w)$, then (2.5) becomes $w^\sigma zw = w^\sigma wz$ and hence $zw = wz$, implying that w commutes with E. But since $\mathrm{Cent}_Q(E) = E$, this implies that $w \in E$, and therefore $s = -N(w) \in -N(E)$.

On the other hand, if $s = N(w)$, then (2.5) becomes $w^\sigma zw = -w^\sigma wz$ and hence $zw = -wz$. This implies that $z \cdot ew = ez^\sigma w = -ezw = ew \cdot z$, and therefore $ew \in \mathrm{Cent}_Q(E) = E$, implying that $w \in eE$. We conclude in this case that $s = N(w) \in N(eE)$. □

Remark 2.5. Even though this result is stated in terms of the subfield E, this subfield is *not* an invariant of the skew-hermitian form h, since it depends on the choice of the first basis vector u, which is completely arbitrary.

2.3 Three related skew-hermitian forms

We will now construct three different skew-hermitian forms, defined over three different quaternion division algebras, which will nevertheless give rise to the same Moufang set. The calculations appearing in this section are of course not very deep, but it is intriguing to observe how everything works out well. They might also be useful for later reference.

So let Q_1, Q_2 and Q_3 be three quaternion division algebras such that $[Q_1] + [Q_2] + [Q_3] = 0$ in the Brauer group $\mathrm{Br}(k)$. By a famous theorem of Albert [8, (16.30)], this implies that these three quaternion algebras have a quadratic subfield E in common (which is *not* necessarily unique). Hence we can find constants $\alpha, \beta, \gamma \in k^*$ such that $Q_1 \cong (E/k, \alpha\beta)$, $Q_2 \cong (E/k, \beta\gamma)$, and $Q_3 \cong (E/k, \alpha\gamma)$. (Of course, two constants would be sufficient to describe this situation, but we prefer to use three constants to have a symmetric description.) As in the previous paragraph, we let e_1 be the element of Q_1 such that $ae_1 = e_1a^\sigma$ for all $a \in E$ and $e_1^2 = \alpha\beta$; similarly we define $e_2 \in Q_2$ and $e_3 \in Q_3$.

Now let $X = E \oplus E \oplus E \oplus E$. We will view X as a vector space over all three quaternion division algebras simultaneously, in such a way that the three scalar multiplications commute. This very peculiar situation can be achieved as follows.

Proposition 2.6. *Let $X = E \oplus E \oplus E \oplus E$, and define the following multiplications on X, for all $t \in E$, and where the e_i are the elements of Q_i (for*

$i=1,2,3$) as above.

$$(a,b,c,d) \bullet_1 t := (at, bt^\sigma, ct, dt)\,; \quad (a,b,c,d) \bullet_1 e_1 := (\alpha\beta b, a, \beta d^\sigma, \alpha c^\sigma)\,;$$

$$(a,b,c,d) \bullet_2 t := (at, bt, ct^\sigma, dt)\,; \quad (a,b,c,d) \bullet_2 e_2 := (\alpha\gamma c, \gamma d^\sigma, a, \alpha b^\sigma)\,;$$

$$(a,b,c,d) \bullet_3 t := (at, bt, ct, dt^\sigma)\,; \quad (a,b,c,d) \bullet_3 e_3 := (\beta\gamma d, \gamma c^\sigma, \beta b^\sigma, a)\,;$$

for all $a,b,c,d \in E$. Then these multiplications extend to scalar multiplications $\bullet_i$ by Q_i (for each i); moreover, $\bullet_i$ and $\bullet_j$ commute when $i \neq j$ (but not *when $i = j$!).*

Proof. It is easily checked that each $\bullet_i$ extends to a scalar multiplication by Q_i (for example, $x \bullet_1 t \bullet_1 e_1 = x \bullet_1 e_1 \bullet_1 t^\sigma$ and $x \bullet_1 e_1 \bullet_1 e_1 = x \bullet_1 \alpha\beta$ for all $x \in X$). On the other hand, an equally easy calculation shows that the scalar multiplications $\bullet_i$ and $\bullet_j$ commute when $i \neq j$. $\square$

Let z be a fixed element of E such that $T(z) = 0$. We now fix our attention on Q_1 and the scalar multiplication $\bullet_1$; we will simply write $\cdot$ in place of $\bullet_1$. Let $u := (1,0,0,0) \in X$ and $v := (0,0,1,0) \in X$; then

$$(a,b,c,d) = u \cdot (a + e_1 b^\sigma) + v \cdot (c + \alpha^{-1} e_1 d)$$

for all $a,b,c,d \in E$. We now define a skew-hermitian form h_1 over Q_1 with trivial discriminant, by the formula (2.4) with $s = -\alpha\gamma$. Note that $N(e_1) = -e_1^2 = -\alpha\beta$. Since Q_3 is a division algebra, $-s = \alpha\gamma \notin N(E)$. Since Q_2 is a division algebra, $sN(\alpha^{-1}e_1) = \beta\gamma \notin N(E)$ and hence $s \notin N(e_1 E)$ either. By Lemma 2.4 therefore, h_1 is anisotropic.

The explicit formulas for the skew-hermitian form h_1 and its corresponding pseudo-quadratic form π_1 are given by

$$\begin{aligned} &h_1((a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2)) \\ &\qquad = z(a_1^\sigma a_2 + \alpha\beta b_1 b_2^\sigma - \alpha\gamma c_1^\sigma c_2 - \beta\gamma d_1^\sigma d_2) \\ &\qquad\qquad - e_1 z(a_1 b_2^\sigma + b_1^\sigma a_2 - \gamma(c_1 d_2 + d_1 c_2))\,; \end{aligned} \tag{2.6}$$

$$\pi_1(a,b,c,d) = z\big(N(a) + \alpha\beta N(b) - \alpha\gamma N(c) - \beta\gamma N(d)\big)/2 - e_1 z(ab^\sigma - \gamma cd)\,.$$

Completely similarly, we can define anisotropic skew-hermitian forms h_2 and h_3 (over Q_2 and Q_3, respectively) and we get

$$\begin{aligned} &h_2((a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2)) \\ &\qquad = z(a_1^\sigma a_2 - \alpha\beta b_1^\sigma b_2 + \alpha\gamma c_1 c_2^\sigma - \beta\gamma d_1^\sigma d_2) \\ &\qquad\qquad - e_2 z(a_1 c_2^\sigma + c_1^\sigma a_2 - \beta(b_1 d_2 + d_1 b_2))\,; \end{aligned}$$

$$\pi_2(a,b,c,d) = z\big(N(a) - \alpha\beta N(b) + \alpha\gamma N(c) - \beta\gamma N(d)\big)/2 - e_2 z(ac^\sigma - \beta bd)\,;$$

$$h_3((a_1,b_1,c_1,d_1),(a_2,b_2,c_2,d_2)) \\ = z(a_1^\sigma a_2 - \alpha\beta b_1^\sigma b_2 - \alpha\gamma c_1^\sigma c_2 + \beta\gamma d_1 d_2^\sigma) \\ - e_3 z(a_1 d_2^\sigma + d_1^\sigma a_2 - \alpha(b_1c_2 + c_1b_2))\,;$$

$$\pi_3(a,b,c,d) = z\big(N(a) - \alpha\beta N(b) - \alpha\gamma N(c) + \beta\gamma N(d)\big)/2 - e_3 z(ad^\sigma - \alpha bc)\,.$$

Already by their construction, these three forms are not unrelated, but it turns out that they have some important related objects in common.

Proposition 2.7. *Let h_1, h_2 and h_3 be the three skew-hermitian forms defined above. Then*

$$N_{Q_1/k}(s+\pi_1(x)) = N_{Q_2/k}(s+\pi_2(x)) = N_{Q_3/k}(s+\pi_3(x))\,; \tag{2.7}$$

$$x \bullet_1 (s+\pi_1(x)) = x \bullet_2 (s+\pi_2(x)) = x \bullet_3 (s+\pi_3(x))\,; \tag{2.8}$$

$$g_1(x,y) = g_2(x,y) = g_3(x,y)\,; \tag{2.9}$$

for all $x,y \in X$ and all $s \in k$. In particular, these three forms define the same Moufang set.

Proof. First, observe that these three forms are defined over different quaternion division algebras, but if we take the norm of their values, then we end up in their common center k, so it makes indeed sense to compare the maps $N_{Q_i/k} \circ \pi_i$. We get

$$\begin{aligned} N(\pi_1(a,b,c,d)) &= N(z)\big(N(a) + \alpha\beta N(b) - \alpha\gamma N(c) - \beta\gamma N(d)\big)^2/4 \\ &\qquad - \alpha\beta N(z) N(ab^\sigma - \gamma cd) \\ &= N(z)/4 \cdot \Big(N(a)^2 + \alpha^2\beta^2 N(b)^2 + \alpha^2\gamma^2 N(c)^2 + \beta^2\gamma^2 N(d)^2 \\ &\qquad - 2\alpha\beta N(a)N(b) - 2\alpha\gamma N(a)N(c) - 2\beta\gamma N(a)N(d) \\ &\qquad - 2\alpha^2\beta\gamma N(b)N(c) - 2\beta^2\alpha\gamma N(b)N(d) \\ &\qquad - 2\gamma^2\alpha\beta N(c)N(d) + 4\alpha\beta\gamma T(ab^\sigma c^\sigma d^\sigma)\Big) \end{aligned}$$

for all $a,b,c,d \in E$. In a similar way, we can calculate $N(\pi_2(a,b,c,d))$ and $N(\pi_3(a,b,c,d))$, and we get the same result, hence $N_{Q_1/k} \circ \pi_1 = N_{Q_2/k} \circ \pi_2 = N_{Q_3/k} \circ \pi_3$. Since $T(\pi(x)) = 0$ for all $x \in X$, it follows, in fact, that

$$N(s+\pi_1(x)) = N(s+\pi_2(x)) = N(s+\pi_3(x))$$

for all $x \in X$ and all $s \in k$, which proves (2.7). By similar calculations, one can check that, for each $i \in \{1,2,3\}$, we get

$$\begin{aligned} &(a,b,c,d) \bullet_i \pi_i(a,b,c,d) \\ &\quad = \big(\; az\big(N(a) - \alpha\beta N(b) - \alpha\gamma N(c) - \beta\gamma N(d)\big)/2 + \alpha\beta\gamma zbcd\,, \\ &\qquad\;\; bz\big(N(a) - \alpha\beta N(b) + \alpha\gamma N(c) + \beta\gamma N(d)\big)/2 - \gamma zac^\sigma d^\sigma\,, \\ &\qquad\;\; cz\big(N(a) + \alpha\beta N(b) - \alpha\gamma N(c) + \beta\gamma N(d)\big)/2 - \beta zab^\sigma d^\sigma\,, \\ &\qquad\;\; dz\big(N(a) + \alpha\beta N(b) + \alpha\gamma N(c) - \beta\gamma N(d)\big)/2 - \alpha zab^\sigma c^\sigma \;\big) \end{aligned}$$

for all $a, b, c, d \in E$; this proves (2.8). Moreover, using (2.2), it is readily checked that

$$g_i((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) \\ = T\big(z(a_1^\sigma a_2 - \alpha\beta b_1^\sigma b_2 - \alpha\gamma c_1^\sigma c_2 - \beta\gamma d_1^\sigma d_2)\big)$$

for all $a_j, b_j, c_j, d_j \in E$ and for each $i \in \{1, 2, 3\}$, and hence (2.9) holds.

Looking back at the construction of the Moufang set in paragraph 2.1, it now follows from the equalities (2.7), (2.8) and (2.9) that the three skew-hermitian forms h_1, h_2 and h_3 give rise to the same Moufang set. □

2.4 The Bartels invariant

It is natural to ask what the deeper connection is between the three skew-hermitian forms of the previous paragraph. It will turn out that they are related by a certain cohomological invariant which has been introduced by Bartels [2]. In its full generality, this invariant is defined on pairs of skew-hermitian forms over a given quaternion division algebra Q which have the same dimension and the same discriminant. We will only need the invariant when one of the two forms is a hyperbolic form (i.e. of maximal Witt index); hence the other form will need to have trivial discriminant.

We will sketch the construction of this invariant in this specific case; see [2] for more details. (Those familiar with cohomological invariants will perhaps find our sketch already too detailed.) So let h be an arbitrary skew-hermitian form over Q with trivial discriminant, and let j a hyperbolic form over Q of the same dimension. Let K denote the separable closure of $k = Z(Q)$. For every finite Galois extension L/k which splits h, there exists a $Q \otimes_k L$-isomorphism φ mapping $h \otimes_k L$ to $j \otimes_k L$. For every $s \in \mathrm{Gal}(L/k)$, let $a_s := \varphi^{-1} \circ {}^s\varphi$; then this defines a cocycle $(a_s) \in H^1(\mathrm{Gal}(L/k), U(j)_L)$, which is independent of the choice of φ. Taking the projective limit over all such extensions, we obtain an element $a(h)$ of $H^1(k, U(j))$, which only depends on the Q-isomorphism class of h. Now consider the exact sequence

$$1 \to SU(j)_K \to U(j)_K \to \mathbb{Z}_2 \to 1$$

which induces the exact Galois cohomology sequence

$$1 \to SU(j)_k \to U(j)_k \to \mathbb{Z}_2 \xrightarrow{\psi} H^1(k, SU(j)) \xrightarrow{\iota} H^1(k, U(j)) \\ \xrightarrow{\delta} H^1(k, \mathbb{Z}_2) \cong k^*/(k^*)^2 \,.$$

Since $\mathrm{disc}(h) = 1 \in k^*/(k^*)^2$, we have $a(h) \in \ker(\delta) = \mathrm{im}(\iota)$. Since Q is a division algebra over k, we have $SU(j)_k = U(j)_k$, and hence ψ is injective. Therefore $\ker(\iota) = \mathrm{im}(\psi) = \mathbb{Z}_2$, and in fact, every fiber of ι contains exactly two elements. In particular, there are exactly two cohomology classes, say with representatives (b_s) and (b'_s), in $H^1(k, SU(j))$ whose image under ι is $a(h)$. Now consider the exact sequence

$$1 \to \mathbb{Z}_2 \to \mathrm{Spin}(j)_K \xrightarrow{\chi} SU(j)_K \to 1$$

(with an appropriate notion of the spin group for skew-hermitian forms). Then for each $s \in \mathrm{Gal}(K/k)$, the elements $b_s, b'_s \in SU(j)_K$ can be lifted to $\mathrm{Spin}(j)_K$ in such a way that the maps $s \mapsto b_s$ and $s \mapsto b'_s$ are still continuous (but of course not cocycles in general). Now let $c_{s,t} := b_s{}^s b_t b_{st}^{-1}$ for all $s, t \in \mathrm{Gal}(K/k)$ and define $c'_{s,t}$ similarly. Then $\chi(c_{s,t}) = 1$ and hence $c_{s,t} \in \mathbb{Z}_2$ for all s, t, and therefore the maps $(s,t) \mapsto c_{s,t}$ and $(s,t) \mapsto c'_{s,t}$ define cocycles in $H^2(k, \mathbb{Z}_2) \cong \mathrm{Br}_2(k)$, the 2-torsion subgroup of $\mathrm{Br}(k)$. It turns out that

$$c + c' = [Q]\,,$$

and hence the pair (c, c') can be seen as an element of $\mathrm{Br}_2(k)/[Q]$. Since the whole construction only depends on the Q-isomorphism class of h, this pair (c, c') is indeed an invariant, which we will denote by $c(h)$.

Proposition 2.8. *Let h be an arbitrary skew-hermitian form over Q with discriminant $\delta := \mathrm{disc}(h)$, let $t \in k^*$ be arbitrary, and let $f := h \perp th$; then* $\mathrm{disc}(f) = 1$.

(i) *If $\delta = 1$, then $c(f) = 0 \mod [Q]$.*

(ii) *If $\delta \neq 1$, then let $E := k(\sqrt{\delta})$ be the discriminant extension field of h. Then $c(f) = [E/k, -t] \mod [Q]$.*

Proof. This follows from [2, Satz 4 (iv),(v)]. □

Using Proposition 2.8, it is not difficult to compute the Bartels invariant $c(h_i)$ for the skew-hermitian forms of the previous section. Let us concentrate on h_1, for example, and recall that h_1 has the form

$$h_1(ux_1 + vy_1, ux_2 + vy_2) = x_1^\sigma z x_2 - \alpha\gamma y_1^\sigma z y_2$$

for all $x_j, y_j \in Q_1$. Let h be the one-dimensional form mapping (ux_1, ux_2) to $x_1^\sigma z x_2$ for all $x_1, x_2 \in Q_1$; then clearly $h_1 \simeq h \perp -\alpha\gamma h$. Also, $\mathrm{disc}(h) = N(z)(k^*)^2$ is non-trivial; in fact, the discriminant extension field is precisely the field $E = \langle 1, z \rangle$. It therefore follows from Proposition 2.8(ii) that

$$\begin{aligned} c(h_1) &= [E/k, \alpha\gamma] \mod [Q_1] \\ &= [Q_3] \mod [Q_1]\,. \end{aligned}$$

Remembering that $[Q_3] + [Q_1] = [Q_2]$, this is equivalent to saying that $c(h_1)$ is in fact equal to the pair $([Q_2], [Q_3])$. The converse also holds:

Proposition 2.9. *Let Q_1, Q_2 and Q_3 be three quaternion division algebras with $[Q_1] + [Q_2] + [Q_3] = 0$ in $\mathrm{Br}(k)$. Up to similarity, h_1 is the only 2-dimensional skew-hermitian form over Q_1 with trivial discriminant and with Bartels invariant $([Q_2], [Q_3])$.*

Proof. We have already computed the Bartels invariant of h_1 above. So let h' be an arbitrary 2-dimensional skew-hermitian form over Q_1 with trivial discriminant and with Bartels invariant $([Q_2], [Q_3])$. By paragraph 2.2, we know that we can write h' in the form

$$h' \simeq h \perp sh$$

where h is the one-dimensional form mapping (ux_1, ux_2) to $x_1^\sigma z x_2$ for all $x_1, x_2 \in Q_1$, as above. It remains to show that $sh \simeq -\alpha\gamma h$. Since $c(h') = [E/K, -s] \mod [Q_1]$, we know that $[E/K, -s] = [Q_2]$ or $[E/K, -s] = [Q_3]$, and therefore $-s\alpha\gamma \in N(E)$ or $-s\beta\gamma \in N(E)$.

Assume first that $-\alpha\gamma = sN(t)$ for some $t \in E$, and let $u' := ut$. Then $sh(u'x_1, u'x_2) = sx_1^\sigma t^\sigma z t x_2 = -\alpha\gamma x_1^\sigma z x_2$, showing that $sh \simeq -\alpha\gamma h$. Assume now that $-\beta\gamma = sN(t)$ for some $t \in E$, and let $u' := ue_1 t\beta^{-1}$. Then $sh(u'x_1, u'x_2) = \beta^{-2} s x_1^\sigma t^\sigma e_1^\sigma z e_1 t x_2 = -\alpha\gamma x_1^\sigma z x_2$, showing again that $sh \simeq -\alpha\gamma h$. $\square$

It is clear that similar results hold when the roles of Q_1, Q_2 and Q_3 are permuted. Hence we have characterized the three skew-hermitian forms h_1, h_2 and h_3 in terms of their defining quaternion algebra and their Bartels invariant only.

Remark 2.10. As Eva Bayer-Fluckiger pointed out to the author, the Bartels invariant for forms with trivial discriminant had already been introduced by J. Tits [14] in a different manner which is also valid in the case $\mathrm{char}(k) = 2$. More precisely, Tits introduces the notion of a *Clifford algebra* $\mathrm{Cl}(\pi)$ corresponding to a pseudo-quadratic form π (over any central simple algebra D of degree d), and shows that if the discriminant is trivial, then $\mathrm{Cl}(\pi)$ decomposes as the direct sum of two simple algebras C_1 and C_2. Moreover, if d is even (in particular if D is a quaternion division algebra), then $[C_1] + [C_2] = [D]$. See [14, Proposition 7].

Remark 2.11. One of the referees cleverly observed that there is another, probably more natural, way to explain the three pairwise commuting multiplications and the connection with the Bartels invariant, in terms of the *Clifford algebra* of the skew-hermitian form (see Remark 2.10!), as follows.

Since $[Q_1] + [Q_2] + [Q_3] = 0$ in $\mathrm{Br}(k)$, we have $Q_1 \otimes Q_2 \otimes Q_3 \cong \mathrm{End}(X)$ for some 8-dimensional vector space X over k. Moreover, there is a non-degenerate alternating bilinear form b on X, uniquely determined up to a scalar, whose adjoint involution is $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$. We can view X as a 2-dimensional vector space over each Q_i, hence

$$Q_2 \otimes Q_3 \cong \mathrm{End}_{Q_1}(X), \quad Q_1 \otimes Q_3 \cong \mathrm{End}_{Q_2}(X), \quad Q_1 \otimes Q_2 \cong \mathrm{End}_{Q_3}(X).$$

So for each i, there is a skew-hermitian form h_i on the Q_i-vector space X, whose adjoint involutions are

$$\sigma_2 \otimes \sigma_3, \quad \sigma_1 \otimes \sigma_3, \quad \sigma_1 \otimes \sigma_2,$$

respectively. Let T_i be the reduced trace form of Q_i; then the transfers $T_i(h_i(x, y))$ are alternating bilinear k-forms whose adjoint involution is $\sigma_1 \otimes \sigma_2 \otimes \sigma_3$, for each i. Hence we may assume, after rescaling, that

$$b(x, y) = T_1(h_1(x, y)) = T_2(h_2(x, y)) = T_3(h_3(x, y))$$

for all $x, y \in X$.

The fact that the adjoint involution of h_1 is $\sigma_2 \otimes \sigma_3$ now yields immediately that the Clifford algebra of h_1 is $Q_2 \times Q_3$, and if both Q_2 and Q_3 are division algebras, then h_1 is anisotropic [8, (15.12), (15.14)].

3 Recovering the structure from the Moufang set

We now take the opposite point of view. We start from the rank one group (or equivalently, from the Moufang set), and we would like to recover as much structure as possible. In view of the results in paragraph 2.3, we cannot expect to recover the complete structure in a unique way, but we hope to be able to recover "triples of related structures". It will turn out that this is indeed possible.

3.1 Properties of the Moufang set

Let h be an anisotropic skew-hermitian form of dimension 2 over a quaternion division algebra Q, with trivial discriminant. In view of Lemma 2.4, we can write h in the form

$$h(ux_1 + vy_1, ux_2 + vy_2) = x_1^\sigma z x_2 + s y_1^\sigma z y_2 \tag{3.1}$$

for all $x_1, x_2, y_1, y_2 \in Q$, with $s \notin -N(E) \cup N(eE)$. Let $\mathbb{M}(h)$ be the Moufang set $\mathbb{M}(U, \tau)$ as in paragraph 2.1, and denote its Hua maps by $\eta_{(a,s)}$ for $(a, s) \in U^*$. (For the definition of these maps, which only depend on U and τ, see [4, 3]; we use η_x in place of h_x to avoid confusion with the skew-hermitian map h. Note that these maps coincide with the "double μ-maps"; see [4, Theorem 3].) By [17, (33.13)] and some calculation involving the isomorphism $U \to T\colon (a, s) \mapsto (a, s + \pi(a))$, we have

$$\eta_{(a,s)}(b, t) = \Big(\big(b - a(s + \pi(a))^{-1} h(a, b)\big) \cdot (s + \pi(a))^\sigma, tN(s + \pi(a))\Big)$$

for all $(a,s) \in U^*$ and all $(b,t) \in U$. Observe that $(X,+) \cong U/Z(U)$ and that each $\eta_{(a,s)}$ normalizes $Z(U)$; hence each $\eta_{(a,s)}$ induces a k-endomorphism of X (which we will continue to denote by $\eta_{(a,s)}$) given by

$$\eta_{(a,s)}(b) = \big(b - a(s+\pi(a))^{-1}h(a,b)\big) \cdot (s+\pi(a))^\sigma \tag{3.2}$$

for all $(a,s) \in U^*$ and all $b \in X$. We now consider the endomorphisms

$$\zeta_{a,s,t} := \eta_{(a,s+t)} - \eta_{(a,s)} - \eta_{(0,t)}$$

with $a \in X^*$, $s \in k$ and $t \in k^*$. Using (3.2), we get

$$\zeta_{a,s,t}(b) = a\Big((s+\pi(a))^{-1}h(a,b)(s+\pi(a))^\sigma - (s+t+\pi(a))^{-1}h(a,b)(s+t+\pi(a))^\sigma\Big)$$

for all $a \in X^*$, $b \in X$, $s \in k$ and $t \in k^*$. Let E_a be the quadratic subfield $\langle 1, \pi(a)\rangle$ of Q, and let e_a be an element of Q orthogonal to E_a (with respect to the trace T of Q). Write $h(a,b) = h_1(a,b) + e_a h_2(a,b)$ with $h_1(a,b), h_2(a,b) \in E_a$. Then

$$\zeta_{a,s,t}(b) = ah_1(a,b)\Big((s+\pi(a))^{-1}(s+\pi(a))^\sigma - (s+t+\pi(a))^{-1}(s+t+\pi(a))^\sigma\Big)$$

and therefore $\zeta_{a,s,t}(b) \in aE_a = \langle a, a\pi(a)\rangle$ for all $a \in X^*$, $s \in k$ and $t \in k^*$. Hence

$$\zeta_{a,k,k^*}(X) = \langle a, a\pi(a)\rangle \tag{3.3}$$

for all $a \in X^*$. These two-dimensional k-subspaces will play a very important role; we will denote them by $R_a := \langle a, a\pi(a)\rangle$ for all $a \in X^*$.

Lemma 3.1. *If $b \in R_a^*$, then $R_b = R_a$.*

Proof. Let $b = as + a\pi(a)t$ for certain $s,t \in k$. We only have to show that $b\pi(b) \in R_a$, and for that, it suffices to show that $\pi(b) \in E_a$. Indeed, $\pi(a\pi(a)) = \pi(a)^\sigma\pi(a)\pi(a) = N(\pi(a))\pi(a) \in E_a$ and $h(a\pi(a),a) = \pi(a)^\sigma h(a,a) = \pi(a)^\sigma 2\pi(a) = 2N(\pi(a)) \in E_a$, and hence $\pi(b) = \pi(as + a\pi(a)) \in E_a$ as claimed. $\square$

Now let $a \in X^*$ be arbitrary and let $b \in X \setminus R_a$. By Lemma 3.1, $R_a \cap R_b = 0$, and hence $L_{a,b} := \langle R_a, R_b\rangle$ is a 4-dimensional k-subspace of X. It turns out that some of these 4-spaces behave nicer than others.

Proposition 3.2. *Let $a \in X^*$ be arbitrary. Then there exist precisely three 4-spaces $L_a^{(i)}$ of the form $L_{a,b}$ such that for each $c \in L_a^{(i)}$, $R_c \subset L_a^{(i)}$. In fact, these three subspaces are the spaces $a \bullet_i Q_i$ for $i \in \{1,2,3\}$.*

Proof. Without loss of generality, we may assume that $a = u$ as in (3.1). If we write $X = E \oplus E \oplus E \oplus E$ as in paragraph 2.3, then $R_a = (E, 0, 0, 0) \subset X$. Let b be an arbitrary element of $X \setminus R_a$, i.e. $b = (q, r, s, t)$ with $q, r, s, t \in E$ and r, s and t not all three equal to zero. Note that $a = (1, 0, 0, 0)$.

Suppose that $R_c \subset L_{a,b}$ for all $c \in L_{a,b}$. Since $L_{a,b}$ is a k-subspace, this implies that $R_{a\lambda+c\mu} \subset L_{a,b}$ for all $c \in L_{a,b}$ and all $\lambda, \mu \in k$ not both zero. One computes that

$$(a\lambda+c\mu)\pi(a\lambda+c\mu) \equiv \lambda\mu\Big(\lambda\big(c\pi(a)+ah(a,c)\big)+\mu\big(a\pi(c)+ch(a,c)\big)\Big) \pmod{L_{a,b}}$$

for all $c \in L_{a,b}$ and all $\lambda, \mu \in k$. Since k is an infinite field, this implies that

$$c\pi(a) + ah(a,c) \in L_{a,b} \tag{3.4}$$

for all $c \in L_{a,b}$.

We now choose $c = (0, r, s, t) \in L_{a,b}$ and we apply the explicit formula (2.6). After some calculation, we obtain that

$$\begin{aligned} c\pi(a) + ah(a,c) &\equiv \big(0, rz/2, sz/2, tz/2\big) && \pmod{R_a} \\ c\pi(c) &\equiv \big(0, rz/2 \cdot \nu_r, sz/2 \cdot \nu_s, tz/2 \cdot \nu_t\big) && \pmod{R_a} \end{aligned}$$

where

$$\begin{aligned} \nu_r &= -\alpha\beta N(r) + \alpha\gamma N(s) + \beta\gamma N(t)\,, \\ \nu_s &= +\alpha\beta N(r) - \alpha\gamma N(s) + \beta\gamma N(t)\,, \\ \nu_t &= +\alpha\beta N(r) + \alpha\gamma N(s) - \beta\gamma N(t)\,. \end{aligned}$$

Since, by (3.4), $c\pi(a) + ah(a,c) \in L_{a,b} = (E, 0, 0, 0) \oplus \langle c, c\pi(c)\rangle$, this implies that there exist elements $\lambda, \mu \in k$ such that

$$\big(0, rz/2 \cdot \nu_r, sz/2 \cdot \nu_s, tz/2 \cdot \nu_t\big) = \big(0, rz/2, sz/2, tz/2\big) \cdot \lambda + (0, r, s, t) \cdot \mu\,;$$

since $\nu_r, \nu_s, \nu_t \in k$ and $z \notin k$, this can only occur if $\mu = 0$. (Recall that r, s and t are not all three equal to zero.)

Now suppose that $r \neq 0$ and $s \neq 0$. Then $\lambda = \nu_r = \nu_s$, implying that $\beta N(r) = \gamma N(s)$ and hence $\beta\gamma = N(\gamma r^{-1} s)$, contradicting the fact that $\beta\gamma \notin N(E)$. It follows that at most one (and hence exactly one) of the elements r, s and t is non-zero. We conclude that $L_{a,b}$ is one of the subspaces $(E, E, 0, 0)$, $(E, 0, E, 0)$ or $(E, 0, 0, E)$. Observe that these spaces are equal to $u \bullet_i Q_i$ for $i \in \{1, 2, 3\}$.

Moreover, keeping in mind that $a\pi(a) = a \bullet_i \pi_i(a)$ is independent of i by (2.8), we get immediately that

$$(a \bullet_i x_i)\pi(a \bullet_i x_i) = a \bullet_i x_i \bullet_i \pi_i(a \bullet_i x_i) \in a \bullet_i Q_i$$

for all $x_i \in Q_i$ and all $i \in \{1, 2, 3\}$, so that the subspaces $L_a^{(i)} := a \bullet_i Q_i$ do indeed satisfy the required condition. □

The collection of all 4-spaces $L_a^{(i)}$ for a fixed i forms an interesting incidence-geometric object. We refer to [7], for example, for definitions of the incidence-geometric notions which we will use.

Proposition 3.3. *For each $i \in \{1,2,3\}$, let*

$$S_i := \{a \bullet_i Q_i \mid a \in X^*\} .$$

Let $\mathbb{X}$ be the 7-dimensional projective space corresponding to X; the elements of S_i are 3-dimensional (projective) subspaces of $\mathbb{X}$. Then S_i is a k-regular spread of $\mathbb{X}$. Moreover, the translation plane arising from the spread S_i is isomorphic to the projective plane over Q_i.

Proof. The last statement is obvious, since we can also view X as a 2-dimensional vector space over Q, which we can also look at as a projective plane over Q; the elements of S are precisely the points of this projective plane. The translation plane arising from this projective plane is this plane itself, which proves the statement.

It is shown in the main theorem (Satz 1) of [6] that the translation plane arising from a spread is regular if and only if the translation plane is Moufang, from which the first statement follows, since a projective plane over a division algebra is Moufang. □

3.2 Recovering the three quaternion algebras

Using the properties of the Moufang set which we have derived in the previous paragraph, we can now reconstruct the quaternion algebras using only the Moufang set $\mathbb{M}(U, \tau)$.

To start with, the additive structure of k can be recognized as the center $Z(U)$ of the group U, and the additive structure of X as the quotient $U/Z(U)$. The maps $\eta_{(a,s)}$ in (3.2) are defined only using the Moufang set (but we have of course no idea yet what the multiplication is or what h and π are). We recognize $(0,1)$ as the only element (a,s) of U such that $\eta_{(a,s)} = 1$. We can define a "multiplication" map from $X \times k$ to X by setting $b \cdot s := \eta_{(0,s)}(b)$ for all $b \in X$ and all $s \in k$; this fits with (3.2). In particular, we have recovered the multiplication on k, since we can put $r = s \cdot t$ to be the only element of k such that $\eta_{(0,t)}\eta_{(0,s)} = \eta_{(0,r)}$, for all $s, t \in k$. Hence X has the structure of a k-vector space (which we know to be 8-dimensional).

By (3.3), we can recognize the subspaces R_a for all $a \in X^*$. By Proposition 3.2, we can recognize the three "nice" subspaces $\{L_a^{(1)}, L_a^{(2)}, L_a^{(3)}\}$ containing R_a (but we cannot distinguish them from each other). However, we know that

$$\{L_a^{(i)} \mid a \in X^*, i \in \{1,2,3\}\} = \{a \bullet_i Q_i \mid a \in X^*, i \in \{1,2,3\}\} .$$

Denote this set of 4-spaces by $\mathcal{F}$; then by Proposition 3.3, $\mathcal{F}$ is the union of three regular spreads. Moreover:

Proposition 3.4. *The only three regular spreads which are contained in $\mathcal{F}$ are S_1, S_2 and S_3.*

Proof. We will call an element of $\mathcal{F}$ of type i if it belongs to S_i. Let S be an arbitrary regular spread contained in $\mathcal{F}$. Let K, L and M be three arbitrary elements of S, and consider the regulus $\mathcal{R}$ through K, L and M. Since k is infinite, there exist a $j \in \{1,2,3\}$ such that at least three elements of $\mathcal{R}$ are of type j. But since S_j is a regular spread, this implies that all elements of $\mathcal{R}$ are of type j, and in particular, K, L and M have the same type. Since K, L and M were arbitrary, we conclude that all elements of S are of the same type, and hence $S = S_j$ for some $j \in \{1,2,3\}$. $\square$

Using Proposition 3.4, we can recognize the three regular spreads $S_i = \{a \bullet_i Q_i \mid a \in X^*\}$. By Proposition 3.3 again, we can reconstruct the projective planes over the quaternion division algebra Q_i for each $i \in \{1,2,3\}$ as the translation planes of these spreads. But this implies that we can recover the three quaternion algebras Q_i (for example as the kernel of the translation planes). More directly, we could also recover Q_i from the spread S_i, as the ring $\{\psi \in \mathrm{End}_k(X) \mid \psi(L) \subseteq L,\ \forall L \in S_i\}$.

To conclude, we have recovered the three quaternion division algebras Q_1, Q_2 and Q_3, but we cannot distinguish between them. We still have to show that for each Q_i, we can recover the pseudo-quadratic form π_i. So fix an i, and let $R_a := a \bullet_i Q_i$ for all $a \in X^*$. Let $a \in X^*$ be arbitrary, and let b be an element in $X \setminus R_a$ such that $\eta_{(b,0)}(a) \in R_a$. By equation (3.2), this can only happen if $h(a,b) = 0$, and hence

$$\eta_{(a,0)}(b) = b \bullet_i \pi_i(a)^{\sigma_i}\,, \tag{3.5}$$

where σ_i is the standard involution of Q_i. Hence $\pi_i(a)$ is the unique element of Q_i such that (3.5) holds. Thus we have recovered the pseudo-quadratic form π_i for each Q_i.

This means that we have indeed recovered the triple of related structures from the Moufang set, as we had claimed in the beginning of this section.

4 The characteristic two case

We now assume that $\mathrm{char}(k) = 2$. In this case, the groups $SU_4(Q,f)$ are sometimes denoted by $SO_4(Q,f)$, since they somehow behave more like quadratic forms (see [13]). In the quaternionic case, which is precisely the case we are dealing with, they have been studied by Seip-Hornix [10, 11]. We can continue to consider the skew-hermitian form h and its corresponding pseudo-quadratic form π as before (but of course, h is now a hermitian form). The main difference is that π is not uniquely determined by h. However, it is still true that h is the unique (skew-)hermitian form such that $h(a,a) = \pi(a) - \pi(a)^\sigma$; see [10, Theorem 1.4]. It is of course no longer true that $T(\pi(a)) = 0$ for all $a \in X$, and neither is equation (2.2) still valid (but equation (2.1) is). Note that π is only determined modulo k; this fact will become more apparent in this case than in the case where $\mathrm{char}(k) \neq 2$. (In fact, we had avoided this issue in the $\mathrm{char}(k) \neq 2$ case by defining π in such a way that $T(\pi(x)) = 0$ for all $x \in X$, but we cannot avoid it in the $\mathrm{char}(k) = 2$ case.)

The group U and the permutation τ which define the Moufang set, now take the following form. Just as in the case $\mathrm{char}(k) \neq 2$, U is the group with underlying set $X \times k$ and with group "addition" given by

$$(a,s) + (b,t) = (a+b, s+t+g(a,b))$$

for all $(a,s),(b,t) \in U$; but now τ is the map from U^* to itself given by

$$\tau\colon (a,s) \mapsto \Big(a(s+\pi(a))^{-1}, \big(s + T(\pi(a))\big) \cdot N(s+\pi(a))^{-1}\Big)$$

for all $(a,s) \in U^*$.

As in section 2.2, we decompose X as $uQ \perp vQ$, hence there exist fixed elements $z, w \in Q$ such that

$$\begin{aligned}\pi(ux+vy) &= x^\sigma z x + y^\sigma w y\,;\\ h(ux_1+vy_1, ux_2+vy_2) &= x_1^\sigma T(z) x_2 + y_1^\sigma T(w) y_2\,;\end{aligned}$$

for all $x, x_1, x_2, y, y_1, y_2 \in Q$. Note that $T(z)$ and $T(w)$ have to be non-zero, since h is non-degenerate. By [11], the (pseudo-)discriminant of π is equal to the class of $N(z)/T(z)^2 + N(w)/T(w)^2$ modulo $\wp(k) := \{x + x^2 \mid x \in k\}$. Since the discriminant is trivial in our case, we have $N(z)/T(z)^2 \equiv N(w)/T(w)^2 \mod \wp(k)$, and it is readily checked that this implies that the fields $\langle 1, z\rangle$ and $\langle 1, w\rangle$ are isomorphic. Hence they are conjugate, and therefore $t^\sigma w t \in \langle 1, z\rangle$ for some $t \in Q$. After a suitable choice of a new orthogonal basis (and keeping in mind that π is only determined modulo k), we can write

$$\begin{aligned}\pi(ux+vy) &= x^\sigma z x + s y^\sigma z^\sigma y\,;\\ h(ux_1+vy_1, ux_2+vy_2) &= x_1^\sigma x_2 + s y_1^\sigma y_2\,;\end{aligned} \tag{4.1}$$

observe that we have chosen z such that $T(z) = 1$, and that we have chosen $w = sz^\sigma$ rather than $w = sz$. Again, we let $E := \langle 1, z\rangle$, and we write $Q = E \oplus eE$. Lemma 2.4 continues to hold: π is anisotropic if and only if $s \in N(e) \cup N(eE)$. Indeed, if $\pi(ux+vy) \in k$, then taking the trace yields

$s \in N(Q)$. Write $s = N(w)$ with $w \in Q$, and let $w = a + eb$ with $a, b \in E$; a little calculation then shows that $ab = 0$ and hence $s \in N(e) \cup N(eE)$.

We are now ready to define again three different pseudo-quadratic forms which will give rise to the same Moufang set. We let $\alpha, \beta, \gamma \in k^*$ and E, Q_1, Q_2, Q_3, e_1, e_2 and e_3 be as in section 2.3. Now let z be a fixed element of E such that $T(z) = 1$. We fix our attention on Q_1 and we will simply write $\cdot$ in place of $\bullet_1$. Let $u := (1, 0, 0, 0) \in X$ and $v := (0, 0, 1, 0) \in X$; then

$$(a, b, c, d) = u \cdot (a + e_1 b^\sigma) + v \cdot (c + \alpha^{-1} e_1 d)$$

for all $a, b, c, d \in E$. We now define a pseudo-quadratic form π_1 and its corresponding hermitian form h_1 over Q_1 with trivial discriminant, by the formulas (4.1) with $s = \alpha\gamma$, and it follows as before that h_1 is anisotropic.

The explicit formulas for the skew-hermitian form h_1 and its corresponding pseudo-quadratic form π_1 are given by

$$\begin{aligned} h_1((a_1, b_1, c_1, d_1), (a_2, b_2, c_2, d_2)) \\ &= (a_1^\sigma a_2 + \alpha\beta b_1 b_2^\sigma + \alpha\gamma c_1^\sigma c_2 + \beta\gamma d_1^\sigma d_2) \\ &\qquad + e_1(a_1 b_2^\sigma + b_1^\sigma a_2 + \gamma(c_1 d_2 + d_1 c_2)) \, ; \end{aligned}$$

$$\pi_1(a, b, c, d) = z\big(N(a) + \alpha\beta N(b)\big) + z^\sigma\big(\alpha\gamma N(c) + \beta\gamma N(d)\big) + e_1(ab^\sigma + \gamma cd) \, .$$

Completely similarly, we can define anisotropic pseudo-quadratic forms π_2 and π_3 with corresponding skew-hermitian forms h_2 and h_3 (over Q_2 and Q_3, respectively); we will omit the explicit formulas here (which should be obvious by now). One can check that $N_{Q_1/k} \circ \pi_1 = N_{Q_2/k} \circ \pi_2 = N_{Q_3/k} \circ \pi_3$ as well as $T_{Q_1/k} \circ \pi_1 = T_{Q_2/k} \circ \pi_2 = T_{Q_3/k} \circ \pi_3$. It follows that equation (2.7) continues to hold in the $\mathrm{char}(k) = 2$ case. A direct calculation also shows that equations (2.8) and (2.9) continue to hold, so we can conclude that the three pseudo-quadratic forms define the same Moufang set, as we claimed.

Let us now take again the other point of view, and start with a given Moufang set as in section 3. Up to some changes in the calculations, the whole argument goes through, and we get the same result that we can recover the three quaternion division algebras Q_1, Q_2 and Q_3, but that we cannot distinguish between them, and that for each Q_i, there is a unique pseudo-quadratic form π_i with corresponding hermitian form h_i which induces the given Moufang set.

Let us point out the main technical difference, which arises from equation (3.4). In the $\mathrm{char}(k) = 2$ case, we obtain from this equation, with the same choice $c = (0, r, s, t) \in L_{a,b}$, that

$$\begin{aligned} c\pi(a) + ah(a, c) &\equiv \big(0, rz, sz, tz\big) && \pmod{R_a} \\ c\pi(c) &\equiv \big(0, r \cdot \nu_r, s \cdot \nu_s, t \cdot \nu_t\big) && \pmod{R_a} \end{aligned}$$

where

$$\begin{aligned}
\nu_r &= z^\sigma \alpha\beta N(r) + z\alpha\gamma N(s) + z\beta\gamma N(t)\,, \\
\nu_s &= z\alpha\beta N(r) + z^\sigma \alpha\gamma N(s) + z\beta\gamma N(t)\,, \\
\nu_t &= z\alpha\beta N(r) + z\alpha\gamma N(s) + z^\sigma \beta\gamma N(t)\,.
\end{aligned}$$

Since, by (3.4), $c\pi(a) + ah(a,c) \in L_{a,b} = (E,0,0,0) \oplus \langle c, c\pi(c)\rangle$, this implies that there exist elements $\lambda, \mu \in k$ such that

$$\big(0, r\cdot\nu_r, s\cdot\nu_s, t\cdot\nu_t\big) = \big(0, rz, sz, tz\big)\cdot\lambda + (0,r,s,t)\cdot\mu\,,$$

which can be rewritten as

$$\big(0, r\cdot(\nu_r + \lambda z + \mu), s\cdot(\nu_s + \lambda z + \mu), t\cdot(\nu_t + \lambda z + \mu)\big) = 0\,.$$

Now suppose that $r \neq 0$ and $s \neq 0$. Then $\lambda z + \mu = \nu_r = \nu_s$, implying that $\beta N(r) = \gamma N(s)$ and hence $\beta\gamma = N(\gamma r^{-1} s)$, contradicting the fact that $\beta\gamma \notin N(E)$. It follows that at most one (and hence exactly one) of the elements r, s and t is non-zero. The rest of the proof is as in the $\mathrm{char}(k) \neq 2$ case.

5 The rank one groups of type ${}^2D_4^{(2)}$

It is natural to ask in how far the theory developed in the main text still applies to the rank one groups of type ${}^2D_4^{(2)}$. These groups are defined in the same way as those of type ${}^1D_4^{(2)}$, the only difference being the fact that the defining skew-hermitian form now has non-trivial discriminant. In this case, we expect to be able to recover the defining quaternion division algebra uniquely from the Moufang set, and this turns out to be so. This can be shown in a completely similar manner as in the ${}^1D_4^{(2)}$ case, and we omit the details.

Notice that there is no need for an analogue of Lemma 2.4, since an arbitrary 2-dimensional skew-hermitian form over a quaternion division algebra Q with non-trivial discriminant is always anisotropic.

Keeping the notation of the main text, we can recover the defining quaternion division algebra (which is now unique!) from the Moufang set, using the same technique.

Proposition 5.1. *Let $a \in X^*$ be arbitrary. Then there exist precisely one 4-space $L_a^{(i)}$ of the form $L_{a,b}$ such that for each $c \in L_a^{(i)}$, $R_c \subset L_a^{(i)}$. In fact, this is the subspace aQ.*

The rest of the process of recovering the structure from the Moufang set can be copied without any change.

References

[1] P. Abramenko and K. S. Brown: *Approaches to buildings*, Book in preparation. Based on *Buildings* by K. S. Brown, Springer-Verlag, 1989.

[2] H.-J. Bartels: "Invarianten hermitescher Formen über Schiefkörpern," *Math. Ann.* **215** (1975), 269–288.

[3] T. De Medts and Y. Segev: "A course on Moufang sets", *Innov. Incidence Geom.* **9** (2009), 79–122.

[4] T. De Medts and R. M. Weiss: "Moufang sets and Jordan division algebras", *Math. Ann.* **335** (2006), no. 2, 415–433.

[5] T. De Medts and H. Van Maldeghem: "Moufang sets of type F_4", submitted.

[6] A. Herzer: *Charakterisierung regulärer Faserungen durch Schliessungssätze*, Arch. Math. **25** (1974), 662–672.

[7] N. Knarr: *Translation Planes*, Lecture Notes in Mathematics **1611**, Springer-Verlag, Berlin, Heidelberg, 1995.

[8] M.-A. Knus, A. S. Merkurjev, H. M. Rost and J.-P. Tignol: *The Book of Involutions*, Amer. Math. Soc. Colloquium Series, Providence, 1998.

[9] W. Scharlau: *Quadratic and Hermitian Forms*, Grundlehren der mathematischen Wissenschaften **270**, Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985.

[10] E. A. M. Seip-Hornix: "Clifford algebras of quadratic quaternion forms. I," *Nederl. Akad. Wetensch. Proc. Ser. A* **68** = Indag. Math. **27** (1965), 326–344.

[11] E. A. M. Seip-Hornix: "Clifford algebras of quadratic quaternion forms. II," *Nederl. Akad. Wetensch. Proc. Ser. A* **68** = Indag. Math. **27** (1965), 345–363.

[12] M. Selbach: *Klassifikationstheorie halbeinfacher algebraischer Gruppen*, Bonner Mathematische Schriften **83**, Bonn, 1976.

[13] J. Tits: Classification of algebraic semisimple groups, *in* "Algebraic Groups and Discontinuous Groups," Boulder 1965, *Proc. Symp. Pure Math.* **9** (1966), 33–62.

[14] J. Tits: *Formes quadratiques, groupes orthogonaux et algèbres de Clifford*, Inventiones Math. **5** (1968), 19–41.

[15] J. Tits: *Buildings of Spherical Type and Finite BN-Pairs*, Lecture Notes in Mathematics **386**, Springer-Verlag, New York/Heidelberg/Berlin, 1974.

[16] J. Tits: *Twin buildings and groups of Kac-Moody type*, in *Groups, combinatorics & geometry (Durham, 1990)*, 249–286, London Math. Soc. Lecture Note Ser. **165**, Cambridge Univ. Press, Cambridge, 1992.

[17] J. Tits and R. M. Weiss: *Moufang Polygons,* Springer Monographs in Mathematics, Springer-Verlag, Berlin, Heidelberg, New York, 2002.

[18] R. Weiss: "The Structure of Spherical Buildings", Princeton University Press, Princeton, 2003.

Tom De Medts, Ghent University, Department of Mathematics, Krijgslaan 281 S23, 9000 Gent, Belgium.

tom.demedts@ugent.be